Curvilinear Coordinates

By

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Introduction

Curvilinear coordinates are local coordinate systems defined on a manifold – a generalization of a surface. A surface is a 2-D manifold. A manifold is defined as a topological space that is Euclidean (flat) a small distance around each point. A differentiable manifold is a manifold where calculus is valid. Cartesian coordinates are considered a global coordinate system because the axes defining the coordinate system remain fixed. In local coordinate systems the axes change based on the position.

Vectors

The concept of vector on a manifold is different than the usual definition because a manifold behaves like a Euclidean coordinate system an infinitesimal distance around each point. The basis vectors of a local coordinate system at each point of a manifold are defined by derivatives because a derivative brings two points an infinitesimal distance from each other to create a tangent vector. Figure 1 shows what would happen if a vector is drawn along the curve – it is an arc with an arrow. This object does not behave as a vector.

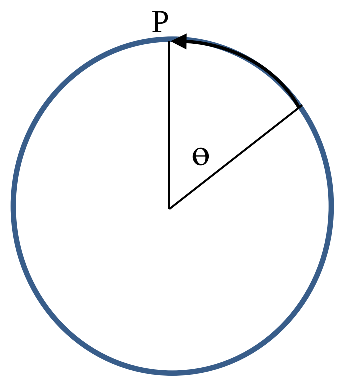


Figure 1

Figure 2 shows how a vector is created by a derivative. The right side is a magnified picture of the left side and shows that a vector is defined by the slope and direction from $f(x)$ to $f(x+h)$.

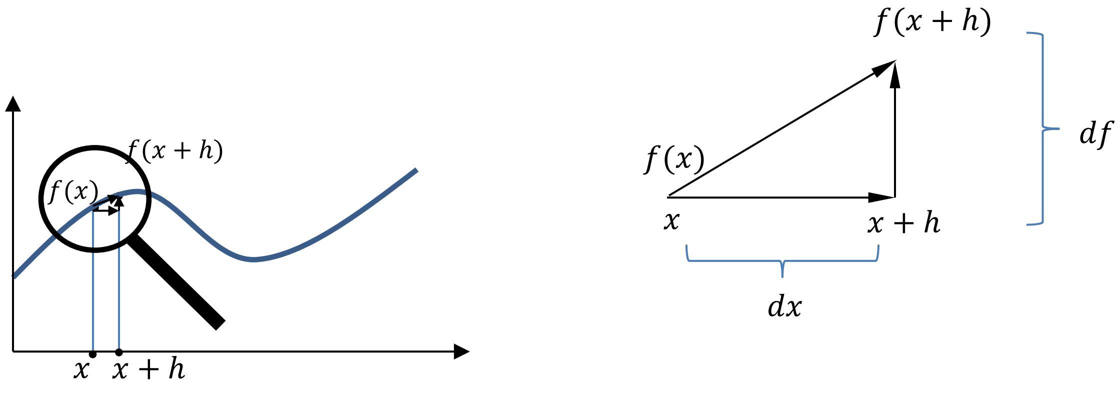


Figure 2

Equation (1) shows the definition of derivative as a limit of the slope as $h \to 0$.

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

So **Tangent Vector to $\boldsymbol{f(x)}$ at x** $\equiv \boldsymbol{\frac{df(x)}{dx}}$

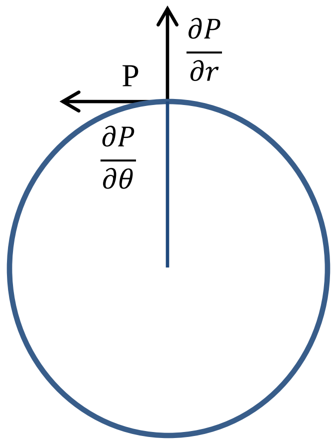


Figure 3

Figure 3 shows that the tangent to the curve at point P and a tangent along the radius are vectors and can be thought of as a coordinate system. Tangent vectors are defined as derivatives as shown above and define local coordinates at P. Since P can be anywhere on the curve, the tangent vectors $\left[\frac{\partial P}{\partial r}, \frac{\partial P}{\partial \theta}\right]$ are defined by equation (2) and are the polar coordinate basis vectors.

$$e_r = \frac{\partial}{\partial r}$$

$$e_\theta = \frac{\partial}{\partial \theta}$$

(2)

Note: when the coordinates such as those in Figure 3 are used, their offset is removed – their origin is moved to the origin in the Cartesian coordinate system.

General Vector Coordinate Transformations

Consider a polar coordinate system.

$$x = rcos(\theta)$$
$$y = rsin(\theta) \tag{3}$$

To find how the basis vectors transform, use the chain rule as follows.

Equation (4) and (5) use the chain rule to define the polar coordinate basis vectors in terms of the Cartesian basis vectors.

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \tag{4}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \tag{5}$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the basis vectors in Cartesian coordinates

Equations (4) and (5) can be viewed in matrix notation as

$$\frac{\partial}{\partial \boldsymbol{Y}} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{Y}}\frac{\partial}{\partial \boldsymbol{X}} \tag{6}$$

where $\boldsymbol{Y} = [r \quad \theta]$, $\boldsymbol{X} = [x \quad y]$ and

$$\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{Y}} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\theta} \end{bmatrix}$$

Equation (6) is nice because $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be any coordinate system.

Expanding equation (4) and (5) from equation (3) gives equations (7).

$$\frac{\partial}{\partial r} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = -rsin(\theta)\frac{\partial}{\partial x} + rcos(\theta)\frac{\partial}{\partial y} \tag{7}$$

The transformation in matrix form is given by equation (8).

$$\bar{e} = A(r,\theta)e = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = A(r,\theta) \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{Y}} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ -\mathrm{rsin}(\theta) & rcos(\theta) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \tag{8}$$

Suppose $\theta = \frac{\pi}{4}$ and $r = 1$, then from (8) the polar coordinate basis functions in terms of the Cartesian basis vectors are

$$\bar{e}_1 = \frac{\partial}{\partial r} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} \tag{9}$$

$$\bar{e}_2 = \frac{\partial}{\partial \theta} = \frac{-1}{\sqrt{2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} \tag{10}$$

Transforming a vector $v = [1 \quad 0]$ to the polar coordinate system $\bar{e}$ is shown below.

$$\bar{v} = A\left(1, \frac{\pi}{4}\right) v = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \tag{11}$$

Multiplying the polar basis functions by the vector values gives

$$\bar{v}_1\bar{e}_1 + \bar{v}_2\bar{e}_2 = \left(1/\sqrt{2}\right)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} - \left(1/\sqrt{2}\right)\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{12}$$

which is what we would expect!

Let's repeat the same process with $r = 2$. The basis functions are now

$$\bar{e}_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} \tag{13}$$

$$\bar{e}_2 = -\sqrt{2}\frac{\partial}{\partial x} + \sqrt{2}\frac{\partial}{\partial y} \tag{14}$$

$$\bar{v} = A\left(2, \frac{\pi}{4}\right) v = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \end{bmatrix} \tag{15}$$

$$\bar{v}_1\bar{e}_1 + \bar{v}_2\bar{e}_2 = \left({}^{1}/_{\sqrt{2}}\right)\begin{bmatrix} {}^{1}/_{\sqrt{2}} \\ {}^{1}/_{\sqrt{2}} \end{bmatrix} - \sqrt{2}\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 2 \\ \frac{1}{2} - 2 \end{bmatrix} = \begin{bmatrix} {}^{5}/_{2} \\ -{}^{3}/_{2} \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{16}$$

Equation (16) shows that in the general case, the vector components don't transform using equation (8).

Box 1

Vector Components Transform Differently Than Their Bases in the General Case

The next step is to find how the vector components transform. Equations (17) and (18) show a general transform for the basis vectors and vector components

$$\bar{e} = Ae \tag{17}$$

$$\bar{v} = Bv \tag{18}$$

The unknown B matrix transforms the components of v from the e to $\bar{e}$ coordinate system.

Since v is fixed in space - coordinate systems are changing – equation (19) must hold. Multiplying the components of $\bar{v}$ out in the transformed coordinate system must produce the components of v in the original coordinate system.

$$\bar{e} \cdot \bar{v} = e \cdot v = v \tag{19}$$

In matrix notation $\bar{e} \cdot \bar{v} \equiv \bar{e}^T\bar{v}$

$$\bar{e}^T\bar{v} = [Ae]^T Bv = e^T A^T Bv = e^T v \tag{20}$$

So

$$A^T B = I \rightarrow B = (A^T)^{-1} = (A^{-1})^T \tag{21}$$

Equation (21) shows that the vector components transform using $B = (A^T)^{-1}$ while the basis vectors transform using A.

Box 2

Basis Vectors Transform Using Matrix $\boldsymbol{A}$

Vector Components Transform Using $\boldsymbol{B} = (\boldsymbol{A^T})^{-1}$

$$B(r,\theta) = [A(r,\theta)^T]^{-1} = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ \frac{-\sin(\theta)}{r} & \frac{cos(\theta)}{r} \end{bmatrix} \quad (22)$$

$$\bar{v} = B\left(2,\frac{\pi}{4}\right)v = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2^{3/2} & 1/2^{3/2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix} \quad (23)$$

$$\bar{v}_1\bar{e}_1 + \bar{v}_2\bar{e}_2 = \left(1/\sqrt{2}\right)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} - \left(1/2^{3/2}\right)\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (24)$$

Now we get the correct answer!

A vector will be denoted as

$$v = v^x e_x + v^y e_y + \cdots \quad (25)$$

where v^i are the vector components and

$$e_i = \frac{\partial}{\partial x^i} \quad (26)$$

We know that the basis vectors e_i transform using $A(r,\theta)$ and the vector components v^i transform using $B(r,\theta)$. Let's try to take the dot product of two vectors by transforming the vector components using $B(r,\theta)$. We know

$$B\left(2,\frac{\pi}{4}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix}$$

We can use $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as the second vector

$$B\left(2,\frac{\pi}{4}\right)\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2^{3/2} & 1/2^{3/2} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix} \quad (27)$$

The dot product now is $\begin{bmatrix} 1/\sqrt{2} & -1/2^{3/2} \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix} = \frac{5}{2} + \frac{1}{8} = \frac{21}{8} \neq \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3$ (28)

So we cannot use $B(r,\theta)$ to transform both vectors – but the vector components transform using $B(r,\theta)$. We need a new type of math construct that gives the correct dot product in the general case.

Box 3

A New Math Construct is Needed to Compute the Inner Product

Equation (29) shows how the components of this new construct transform.

$$\bar{\alpha} = C\alpha \tag{29}$$

We know that

$$\bar{v} = Bv \tag{30}$$

Now equation (31) must hold to keep the dot product the same in all coordinate systems.

$$\bar{\alpha} \cdot \bar{v} = \alpha \cdot v = constant \tag{31}$$

Substituting equations (29) and (30) into equation (31) gives equations (32) and (33).

$$\bar{\alpha}^T \bar{v} = [C\alpha]^T Bv = \alpha v \tag{32}$$

$$\alpha^T C^T Bv = \alpha^T v \tag{33}$$

So

$$C^T B = I$$

$$B = (C^T)^{-1} = (A^T)^{-1}$$

So $\quad C = A$

So the new constructs components transform as

$$\bar{\alpha} = A\alpha \tag{34}$$

To test, transform one of the vectors using $A\left(2, \frac{\pi}{4}\right)$

$$A\left(2,\frac{\pi}{4}\right)\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -\sqrt{2} & \sqrt{2}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}5/\sqrt{2}\\ -\sqrt{2}\end{bmatrix} \tag{35}$$

The dot product now is $\begin{bmatrix}5/\sqrt{2} & -\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ -1/2^{3/2}\end{bmatrix} = 5/2 + 1/2 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = 3$

We get the correct answer!

Box 4

This new construct is called a covector or a differential 1 Form.

Differential 1 Forms

Differential 1-Forms are less known than vectors. They are any expression involving a linear combination of single differentials. For example $Adx + Bdy$ is a differential 1 form. Differential 1 forms in 2-D are lines because $ax + by = c$ is the equation of a line – Adx are lines parallel to the y axis, Bdy are lines parallel to the x axis, $dx + dy$ are lines perpendicular to a line at a 45 degree angle.

In 3-D, differential 1 forms $Adx + Bdy + Cdz$ are visualized as planes because the equation for a plane is $ax + by + cz = d$. In 3-D, Adx are planes parallel to the y-z plane, Bdy are planes parallel to the x-z plane, and Cdz are planes parallel to the x-y plane.

A plane – (line) - can be characterized by a vector normal to the plane – (line) Using normal vectors allows visualization of 1 forms as vectors – see figures 3 and 4 below

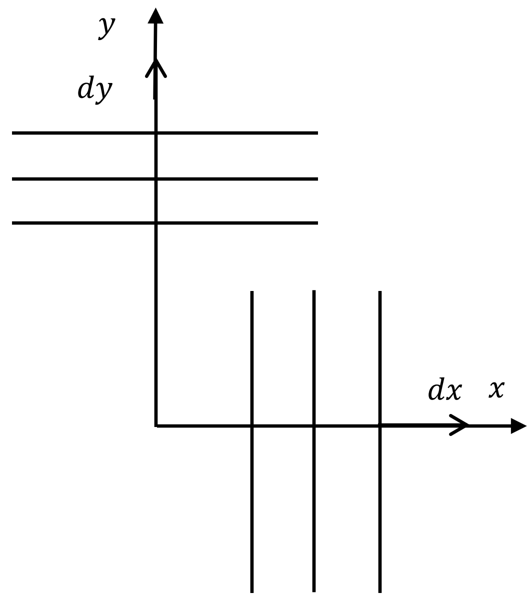


Figure 3

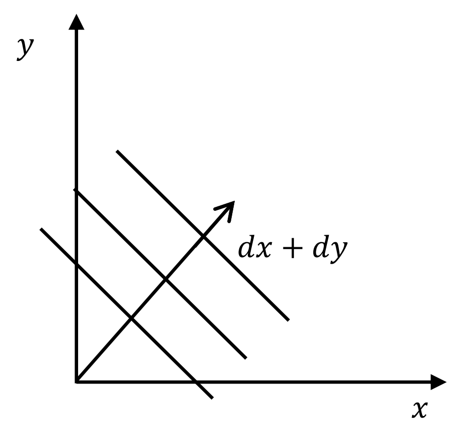


Figure 4

The normal vectors of differential forms show that 1 – Forms add like vectors – see Figure 5.

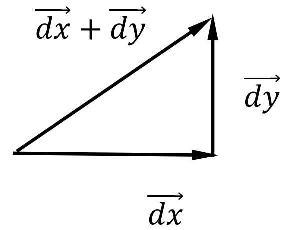


Figure 5

Box 5

Normal Vectors of 1-Forms Show 1-Forms Add Like Vectors

1 forms are numerically evaluated between two vectors. For example, $dx + 2dy + 3dz$ evaluated over
$v1 = [1 \quad 2 \quad 1]$ and $v2 = [3 \quad 6 \quad 5]$ is $(3 - 1) + 2(6 - 2) + 3(5 - 1) = 2 + 8 + 12 = 22$

1 forms are usually used – as differentials - without numerically evaluating them.

In the language of differential forms, the basis 1 forms in polar coordinates are dr and $d\theta$ – see Figure 6.

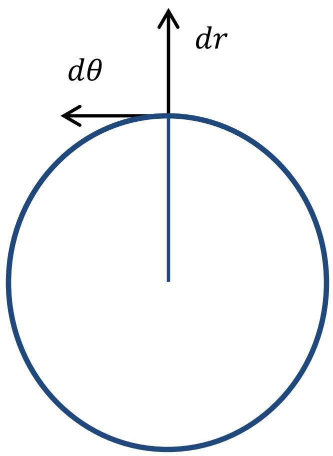


Figure 6

Comparing Figures 2 and 6, the reader may ask what the differences between one forms and vectors are. The difference is that they transform differently.
In the last section, we showed that the coefficients of a differential 1-Form transform like equation (34) – shown again below.

$$\bar{\alpha} = A\alpha \tag{34}$$

To find out how the basis 1-Forms transform, start with equation (34) and (36)

$$\bar{\omega} = D\omega \tag{36}$$

Using the same approach we used earlier

$$\bar{\omega} \cdot \bar{\alpha} = \omega\alpha \rightarrow \bar{\omega}^T \bar{\alpha} = \omega^T \alpha \tag{37}$$

$$\omega^T D^T A\alpha = \omega^T \alpha \rightarrow D = (A^T)^{-1} = B \tag{38}$$

Box 6

1-Form Coordinates Transform Using $\bar{\boldsymbol{\alpha}} = \boldsymbol{A\alpha}$

1-Form Bases Transform using $\bar{\boldsymbol{\omega}} = \boldsymbol{B\omega}$

Equations (39) and (40) show how the vector and 1-Forms basis sets transform.

$$\bar{e} = Ae \tag{39}$$

$$\bar{\omega} = B\omega \tag{40}$$

We can see relationship between the vector and 1-Form basis sets by using the outer product. For the following argument, we set e and ω are in Cartesian Coordinates.

$$\bar{e} \otimes \bar{\omega} \tag{41}$$

In matrix notation

$$\bar{e}\bar{\omega}^T = Ae(B\omega)^T = Ae\omega^T B^T = Ae\omega^T A^{-1} = \tag{42}$$

but

$$e\omega^T = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} [dx \quad dy] = \begin{bmatrix} \dfrac{dx}{dx} & \dfrac{dy}{dx} \\ \dfrac{dx}{dy} & \dfrac{dy}{dy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{43}$$

where

$$\frac{dy}{dx} = \frac{dy}{dx} = 0$$

because x and y are independent of each other

and

$$\frac{dx}{dx} = \frac{dy}{dy} = 1$$

So

$$\bar{e}\bar{\omega}^T = Ae\omega^T A^{-1} = I \qquad (44)$$

The dot product properties of $[dr \quad d\theta]$ and $\left[\frac{\partial}{\partial r} \quad \frac{\partial}{\partial \theta}\right]$ are shown in Table 1.

	dr	$d\theta$
$\frac{\partial}{\partial r}$	$\frac{\partial}{\partial r} \cdot dr = dr \cdot \frac{\partial}{\partial r} = \frac{dr}{dr} = 1$	$\frac{\partial}{\partial r} \cdot d\theta = d\theta \cdot \frac{\partial}{\partial r} = \frac{d\theta}{dr} = 0$
$\frac{\partial}{\partial \theta}$	$\frac{\partial}{\partial \theta} \cdot dr = dr \cdot \frac{\partial}{\partial \theta} = \frac{dr}{d\theta} = 0$	$\frac{\partial}{\partial \theta} \cdot d\theta = d\theta \cdot \frac{\partial}{\partial \theta} = \frac{d\theta}{d\vartheta} = 1$

Table 1

We can use the chain rule on equation (2) to see that we get the matrix $B(r, \theta)$.

$$x = rcos(\theta)$$
$$y = rsin(\theta) \qquad (2)$$

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy \qquad (45)$$

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy \qquad (46)$$

$$r(x, y) = [x^2 + y^2]^{1/2} \qquad (47)$$

$$\theta(x,y) = tan^{-1}\left(\frac{y}{x}\right) \tag{48}$$

$$dr = \cos(\theta)\, dx + \sin(\theta)\, dy \tag{49}$$

$$d\theta = \frac{-1}{r}\sin(\theta)\, dx + \frac{1}{r}\cos(\theta) dy \tag{50}$$

Equation (51) shows equations (49) and (50) in matrix form.

$$\begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ \frac{-\sin(\theta)}{r} & \frac{cos(\theta)}{r} \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = B(r,\theta) \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{51}$$

The generalization of equations (45) and (46) can be represented using the matrix equation (52).

$$d\boldsymbol{Y} = \left[\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}}\right] d\boldsymbol{X} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{bmatrix} \tag{52}$$

Comparing equation (52) with equations (45) and (46)

$$B(r,\theta) = \left[\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}}\right]^T$$

So

$$\left[\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{X}}\right]^T \left[\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{Y}}\right] = I \tag{53}$$

A 1-Form will be denoted as

$$\alpha = \alpha_x \omega^x + \alpha_y \omega^y + \cdots \tag{54}$$

where α_i are the 1-form components and $\omega^i = dx^i$ are the 1-Form basis.

Summary of Coordinate Transform Relationships

There are two types of geometric objects.

1.) Vectors
2.) 1 forms

Table 2 shows the coordinate transform relationships for vectors and 1-Forms.

	Basis	Components
Vector	$\bar{e} = Ae$	$\bar{\alpha} = A\alpha$
1 Form	$\bar{\omega} = B\omega$	$\bar{v} = Bv$

Table 2

Notation

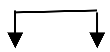
There are several types of notation used in this write up. The first one is matrix notation. Most of the notation used so far is matrix notation. The advantage of this notation is that the form of the equations stay the same regardless of the components. The second one is an indexed matrix notation where matrix elements are referenced by indices. In this notation all the indices are in the lower index position. The third notation is Einstein notation where upper and lower indices indicate how an object transforms. The reason we include the matrix indexing is that it helps to translate from matrix equations to Einstein indexing. These ideas will become clearer in the following discussion.

Equation (55) shows a dot product between vectors in matrix and indexed matrix notation. A repeated index indicates that the indices should be summed over.

$$v^T e = \begin{bmatrix} v_1 & v_2 & \cdots \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \end{bmatrix} = v_1 e_1 + v_2 e_2 + \cdots = v_i e_i \tag{55}$$

Einstein notation helps keep track of vector and 1-Form basis sets and components. Vector and one form basis sets are shown in equation (56).

$$e_i = \frac{\partial}{\partial x^i}$$



$$\omega^i = dx^i \tag{56}$$

Notice that the position – upper or lower – on the left side corresponds to the position of the differential on the right side. Upper on the left corresponds to dx^i in the numerator and lower on the left corresponds to ∂x^i in the denominator. In general, a component that transforms like a basis vector has a lower index position and a component that transforms like a 1-Form basis vector has an upper index. Using table 2, vector components transform using an upper index and 1-Form components transform using a lower index.

A second rule is the Einstein summation convention. An expression that has an upper index component and a lower index component with the same index - is summed over. This is called a bound index. For example, equation (25) can be rewritten as

$$v^i e_i = v^x e_x + v^y e_y + \cdots \tag{56}$$

Equation (54) can be rewritten as

$$\alpha_i \omega^i = \alpha_x \omega^x + \alpha_y \omega^y + \cdots \tag{57}$$

Indices can also be used as free indices as shown in equation (58).

$$v^i = \begin{bmatrix} v^x \\ v^y \\ \vdots \end{bmatrix} \tag{58}$$

Note that (56) and (57) are inner products. If the index was not the same in the upper and lower position, it would be an outer product.

For $i = 1,2 \quad j = 1,2$

$$v^i e_j = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} v^1 e_1 & v^1 e_2 \\ v^2 e_1 & v^2 e_2 \end{bmatrix} \tag{59}$$

Equation (44) can be expressed as

$$\bar{e}\bar{\omega}^T = e\omega^T = \frac{d\bar{e}_i}{d\bar{e}_j} = \delta^i_j = \begin{cases} 0 \; i \neq j \\ 1 \; i = j \end{cases}$$

This result holds in any coordinate system.

We could have both indices in the upper or lower also. A matrix can be described by

$$G = g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

and its inverse can be described by

$$G^{-1} = g^{ij} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}$$

(60)

A mixed index construct can be described by

$$\left[T^i{}_j\right]_{ij} = \begin{bmatrix} T^1{}_1 & T^1{}_2 & \cdots \\ T^2{}_1 & T^2{}_2 & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

(61)

a matrix multiplied by a vector can be described by

$$Gv = G_{ij}v_j = g_{ij}v^j$$

(62)

We will figure out how to use Einstein indexing on the expression AA^T. Follow the progression in equation (63).

1.) Start with definition of A matrix

$$A = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \cdots \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

2.) Associate matrix A in matrix index form with Einstein form

$$A_{ij} = \frac{\partial x^j}{\partial \bar{x}^i}$$

3.) Use definition of A^T matrix

$$A^T = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \cdots \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

4.) Associate matrix A^T in matrix index form with Einstein form

$$A^T{}_{ij} = \frac{\partial x^i}{\partial \bar{x}^j}$$

Note: we interchanged the upper and lower indices

5.) Translate matrix multiplication using matrix index form to Einstein form

$$[AA^T]_{ij} = A_{in}A^T{}_{nj} = \frac{\partial x^n}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^j} \tag{61}$$

Clearly, equation (61) is missing something because we want to sum over the n index but the n indices are both in the upper index. This problem can be solved by adding the identity matrix into the equation - set $I = \delta_{ij}$

$$[AIA^T]_{ij} = A_{in}\delta_{nn}A^T{}_{nj} = \frac{\partial x^n}{\partial \bar{x}^i}\delta_{nn}\frac{\partial x^n}{\partial \bar{x}^j} = g_{ij} \tag{62}$$

Now the equation sums over both the upper indices n and is correct!

If something has three indices, we can represent it as a block vector with each element a matrix.

$$T^i{}_{jk} = \left[\left[T^1{}_{jk}\right] \quad \left[T^2{}_{jk}\right]\right] \tag{63}$$

Figure 7 shows a diagram depicting the relationship between the vector and 1-Form basis sets and coefficients. The right side of the diagram – Nodes 1 and 2 are in Cartesian Coordinates. Nodes 3 and 4 are the transformed – new -coordinate system.

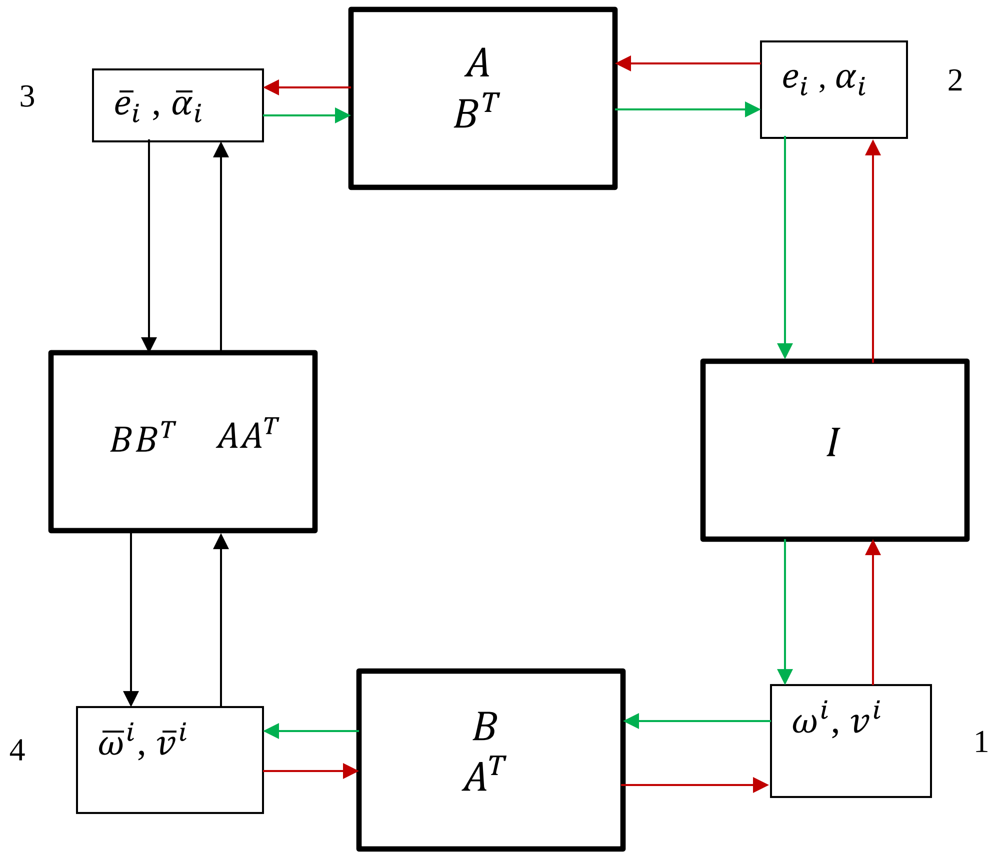


Figure 7

To transform a lower coordinate to an upper coordinate in the new system, start on node 3 and follow the green arrows. The unknown is the path from Node 3 to Node 4. First multiply by B^T to get to Node 2, then multiply by I (because the system is Cartesian coordinates) to get to Node 1, and finally multiply by$B$ to get to Node 4. This process is shown in equation (64).

$e_i = B^T \bar{e}_i$ Node 3 to Node 2

$\omega^i = I e_i$ Node 2 to Node 1 (Cartesian Coordinates)

$\bar{\omega}^i = B\omega^i$ Node 1 to Node 4

$\bar{\omega}^i = B\omega^i = BIe_i = BIB^T\bar{e}_i = BB^T\bar{e}_i$ Node 3 to Node 4

(64)

Equation (65) shows the steps to transform from matrix to Einstein notation.

1.) Write out components of the B matrix

$$B = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

2.) Write out matrix elements of B in index form

$$B_{ij} = \frac{\partial \bar{x}^i}{\partial x^j}$$

3.) Write out matrix elements of B^T in index form by interchanging the indices

$$B^T{}_{ij} = \frac{\partial \bar{x}^j}{\partial x^i}$$

4.) Write out expression in matrix index form and then translate it to Einstein notation

$$\bar{\omega}^i = [BIB^T]_{ij}\bar{e}_j = \left[B_{in}\delta_{nn}B^T{}_{nj}\right]\bar{e}_j = \frac{\partial \bar{x}^i}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^j}{\partial x^n}\bar{e}_j = g^{ij}\bar{e}_j$$

(65)

Equation (65) shows matrix index notation i.e. $[BIB^T]_{ij}\bar{e}_j$ and describes a matrix multiplication in index form as discussed above. The repeated index is summed over. The indices are all lower indices in matrix index form. Using matrix indexing provides an intermediate step between matrix notation and Einstein notation.

Let's transform $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$using $A\left(2, \frac{\pi}{4}\right)$ which creates a lower index coefficient $\bar{\alpha}_i$

$$\bar{\alpha}_i = A\left(2, \frac{\pi}{4}\right)\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

Now use

$$B(r,\theta)IB(r,\theta)^T = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ \frac{-\sin(\theta)}{r} & \frac{cos(\theta)}{r} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \frac{-\sin(\theta)}{r} \\ \sin(\theta) & \frac{cos(\theta)}{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}$$

to go from a lower index $\bar{\alpha}_i$ to upper index $\bar{v}^i$

with $r = 2$

$$BIB^T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\bar{v}_i = \bar{v} = BIB^T\bar{\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -2^{1/2-2} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix} = B\left(2, \frac{\pi}{4}\right) \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

See equation (27).

$$B\left(2, \frac{\pi}{4}\right) \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2^{3/2} & 1/2^{3/2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix} \quad (27)$$

These are upper components because we used the B matrix for the transformation. The equations check out!

Now, we can check equation (65) – Einstein notation version - to make sure we get the same answer.

$$B = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$B\left(2, \frac{\pi}{4}\right) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2^{3/2} & 1/2^{3/2} \end{bmatrix}$$

$$\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^n} \delta^{nn} \frac{\partial \bar{x}^j}{\partial x^n} \bar{\alpha}_j$$

$$\bar{v}^1 = \frac{\partial \bar{x}^1}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^j}{\partial x^n}\bar{e}_j = \frac{\partial \bar{x}^1}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^1}{\partial x^n}\bar{e}_1 + \frac{\partial \bar{x}^1}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^2}{\partial x^n}\bar{e}_2$$

$$= \left[\frac{\partial \bar{x}^1}{\partial x^1}\frac{\partial \bar{x}^1}{\partial x^1} + \frac{\partial \bar{x}^1}{\partial x^2}\frac{\partial \bar{x}^1}{\partial x^2}\right]\bar{e}_1 + \left[\frac{\partial \bar{x}^1}{\partial x^1}\frac{\partial \bar{x}^2}{\partial x^1} + \frac{\partial \bar{x}^1}{\partial x^2}\frac{\partial \bar{x}^2}{\partial x^2}\right]\bar{e}_2$$

$$= \left[\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right]\bar{e}_1 + \left[-\frac{1}{\sqrt{2}}\frac{1}{2^{3/2}} + \frac{1}{\sqrt{2}}\frac{1}{2^{3/2}}\right]\bar{e}_2 = 1\cdot\bar{e}_1 + 0\cdot\bar{e}_2 = {}^{5}/_{\sqrt{2}}$$

$$\bar{v}^2 = \frac{\partial \bar{x}^2}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^j}{\partial x^n}\bar{e}_j = \frac{\partial \bar{x}^2}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^1}{\partial x^n}\bar{e}_1 + \frac{\partial \bar{x}^2}{\partial x^n}\delta^{nn}\frac{\partial \bar{x}^2}{\partial x^n}\bar{e}_2$$

$$= \left[\frac{\partial \bar{x}^2}{\partial x^1}\frac{\partial \bar{x}^1}{\partial x^1} + \frac{\partial \bar{x}^2}{\partial x^2}\frac{\partial \bar{x}^1}{\partial x^2}\right]\bar{e}_1 + \left[\frac{\partial \bar{x}^2}{\partial x^1}\frac{\partial \bar{x}^2}{\partial x^1} + \frac{\partial \bar{x}^1}{\partial x^2}\frac{\partial \bar{x}^2}{\partial x^2}\right]\bar{e}_2$$

$$= \left[-\frac{1}{2^{3/2}}\frac{1}{\sqrt{2}} + \frac{1}{2^{3/2}}\frac{1}{\sqrt{2}}\right]\bar{e}_1 + \left[\frac{1}{2^{3/2}}\frac{1}{2^{3/2}} + \frac{1}{2^{3/2}}\frac{1}{2^{3/2}}\right]\bar{e}_2 = 0\cdot\bar{e}_1 + \left[\frac{1}{8} + \frac{1}{8}\right]\bar{e}_2$$

$$= \frac{1}{4}\bar{e}_2 = \frac{1}{4}\left(-\sqrt{2}\right) = -\frac{\sqrt{2}}{4} = -2^{1/2-2} = -2^{-3/2} = -\frac{1}{2^{3/2}}$$

So $\bar{v} = \begin{bmatrix} {}^{5}/_{\sqrt{2}} \\ -{}^{1}/_{2^{3/2}} \end{bmatrix}$ - which is the correct answer!

To transform from an upper coordinate to a lower coordinate in the new coordinate system, start on Node 4 – Figure 7 - and follow the red arrows. The unknown is the path from Node 4 to Node 3. First multiply by A^T to get to Node 1, multiply by I to get to Node 2, and then multiply by A to get to Node 3 as shown in equation (66).

$\omega^i = A^T\bar{\omega}^i$ Node 4 to Node 1

$e_i = I\omega^i$ Node 1 to Node 2

$\bar{e}_i = Ae_i = AI\omega^i = AIA^T\bar{\omega}^i = AIA^T\bar{\omega}^i$ Node 2 to Node 3

(66)

Equation (67) shows equation (66) in Einstein notation – see equation (62).

$$\bar{e}_j = \frac{\partial x^n}{\partial \bar{x}^i}\delta_{nn}\frac{\partial x^n}{\partial \bar{x}^j}\bar{\omega}^i = g_{ij}\bar{\omega}^i$$

(67)

Now use $AIA^T = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$ to transform an upper index to a lower index.

With $r = 2$

$$\bar{\alpha} = AIA^T \bar{\omega} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} \\ -1/2^{3/2} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -2^{-3/2+2} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ -\sqrt{2} \end{bmatrix} = A\left(2, \frac{\pi}{4}\right) \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Now, we can check equation (67) to make sure we get the same answer.

$$A = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \cdots \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$A\left(2, \frac{\pi}{4}\right) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$\bar{\alpha}_1 = \frac{\partial x^n}{\partial \bar{x}^i} \delta_{nn} \frac{\partial x^n}{\partial \bar{x}^1} \bar{\omega}^i = \frac{\partial x^n}{\partial \bar{x}^1} \delta_{nn} \frac{\partial x^n}{\partial \bar{x}^1} \bar{\omega}^1 + \frac{\partial x^n}{\partial \bar{x}^2} \delta_{nn} \frac{\partial x^n}{\partial \bar{x}^1} \bar{\omega}^2$$

$$= \left[\frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^1} + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^1}\right] \bar{\omega}^1 + \left[\frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^1}{\partial \bar{x}^1} + \frac{\partial x^2}{\partial \bar{x}^2}\frac{\partial x^2}{\partial \bar{x}^1}\right] \bar{\omega}^2$$

$$= \left[\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right] \frac{5}{\sqrt{2}} + \left[-\sqrt{2}\frac{1}{\sqrt{2}} + \sqrt{2}\frac{1}{\sqrt{2}}\right]\left(-\frac{1}{2^{3/2}}\right) = \frac{5}{\sqrt{2}} + 0 = \frac{5}{\sqrt{2}}$$

$$\bar{\alpha}_2 = \frac{\partial x^n}{\partial \bar{x}^i} \delta_{nn} \frac{\partial x^n}{\partial \bar{x}^2} \bar{\omega}^i = \frac{\partial x^n}{\partial \bar{x}^1} \delta_{nn} \frac{\partial x^n}{\partial \bar{x}^2} \bar{\omega}^1 + \frac{\partial x^n}{\partial \bar{x}^2} \delta_{nn} \frac{\partial x^n}{\partial \bar{x}^2} \bar{\omega}^2$$

$$= \left[\frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^1}{\partial \bar{x}^2} + \frac{\partial x^2}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2}\right] \bar{\omega}^1 + \left[\frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^1}{\partial \bar{x}^2} + \frac{\partial x^2}{\partial \bar{x}^2}\frac{\partial x^2}{\partial \bar{x}^2}\right] \bar{\omega}^2$$

$$= \left[-\frac{1}{\sqrt{2}}\sqrt{2} + \frac{1}{\sqrt{2}}\sqrt{2}\right] \bar{\omega}^1 + \left[\sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{2}\right]\bar{\omega}^2$$

$$0 \cdot \bar{\omega}^1 + 4 \cdot \bar{\omega}^2 = {}^{-4 \cdot 1}/_{2^{3/2}} = -\left[2^{2-3/2}\right] = -\sqrt{2}$$

The result is $\bar{\alpha} = \begin{bmatrix} 5/\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$ and is the correct answer!

Both AIA^T and BIB^T are symmetric because a matrix multiplied by the transpose of itself creates a symmetric matrix - $AIA^T \equiv g_{ij} = g_{ji}$, $BIB^T \equiv g^{ij} = g^{ji}$ Both AIA^T and BIB^T are special matrices - $\boldsymbol{AIA^T}$ is called the metric and $\boldsymbol{BIB^T}$ is called the inverse metric. The metric is the topic of the next section.

Box 7

Matrix Notation - $\boldsymbol{AIA^T}$

Matrix Index Notation - $\boldsymbol{A_{ij}}$ **- -** $\boldsymbol{AIA^T = A_{in}I_{nn}A^T{}_{nj}}$

Einstein Notation

$$\boldsymbol{e_i = \frac{\partial}{\partial x^i}} \qquad \boldsymbol{\omega^i = dx^i}$$

$$\boldsymbol{a_i b^i = a_1 b^1 + a_2 b^2 + \cdots}$$

The Metric

The concept of distance in a coordinate system is given by the differential path length. Equation (69) shows this path length in Cartesian coordinates.

$$ds^2 = dx^2 + dy^2 + \cdots \tag{69}$$

In vector form

$$ds^2 = \omega^T \omega \tag{70}$$

where $\omega^T = [dx \quad dy \quad \cdots]$

From Figure 7 – Node 4 to Node 1

$$\omega = A^T \bar{\omega} \tag{71}$$

so

$$ds^2 = \omega^T I \omega = [A^T \overline{\omega}]^T I A^T \overline{\omega} = \overline{\omega}^T A I A^T \overline{\omega} = \overline{\omega}^T G \overline{\omega} \qquad (72)$$

$G = AIA^T$ is called the metric because it preserves the path length in the new coordinate system. The path length is always equivalent to the Cartesian path length.

Notice that equation (72) shows the inner product across coordinate systems. Vector coordinates transform like 1-Form bases, so equation (72) defines an inner product of vectors using their components.

$$v^T I v = \bar{v}^T G \bar{v}$$

For two different vector components - v_1 and v_2

$$v_1{}^T I v_2 = \bar{v}_1{}^T G \bar{v}_2$$

Also in Figure 7 we can go from Node 4 to Node 2

$$e = A^T I \overline{\omega}$$

We effectively lowered the index because in Cartesian coordinates the vector and 1-Form bases point in the same direction.

$$\bar{e} = Ae$$

So

$$\bar{e} = Ae = AIA^T \overline{\omega} = G \overline{\omega}$$

which lowers the index! – this is how we constructed the metric in the previous section.

Equation (72) also shows $\omega^i \delta_{ij} \omega^j = \overline{\omega}^i g_{ij} \overline{\omega}^j$

So

δ_{ij}in Cartesian coordinates transforms to g_{ij} in the new coordinate system.

In the polar coordinate system

$$G = AIA^T = \begin{bmatrix} \cos(\theta) & sin(\theta) \\ -\mathrm{r}\sin(\theta) & rcos(\theta) \end{bmatrix} I \begin{bmatrix} \cos(\theta) & -rsin(\theta) \\ \sin(\theta) & rcos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

so

$$ds^2 = \overline{\omega}^T G \overline{\omega} = [dr \quad d\theta] \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix} = dr^2 + r^2 d\theta^2$$

$$= [cos(\theta)dx + sin(\theta)dy]^2 + r^2 \left[-\frac{1}{r} sin(\theta)dx + \frac{1}{r} cos(\theta)dy\right]^2$$

$$= cos^2(\theta)dx^2 + 2cos(\theta)sin(\theta)dxdy + sin^2(\theta)dy^2$$

$$+sin^2(\theta)dx^2 - 2sin(\theta)cos(\theta)dxdy + cos^2(\theta)dy^2$$

$$= [cos^2(\theta) + sin^2(\theta)]dx^2 + [cos^2(\theta) + sin^2(\theta)]dy^2 = dx^2 + dy^2$$

(73)

Equation (73) in Einstein notation is written as

$$ds^2 = g_{ij} dx^i dx^j \quad (74)$$

Where the units are m^2

There is an inverse version of the path length.

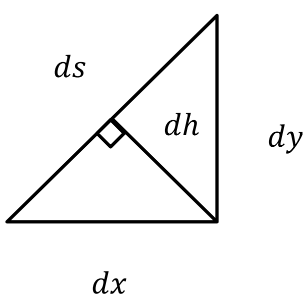


Figure 8

Figure 8 shows a right triangle with height dh perpendicular to side ds.

There are two relations implied in Figure 8.

1.) Pythagorean Theorem

$$ds^2 = dx^2 + dy^2$$

and

2.) Pythagorean Theorem for Reciprocals

$$\left[\frac{\partial}{\partial h}\right]^2 = \left[\frac{\partial}{\partial x}\right]^2 + \left[\frac{\partial}{\partial y}\right]^2 \text{[1]} \quad (74)$$

where dh is the height as shown in Figure 8.

For example if $a = 3$, $b = 4$, and $c = 5$

Then Area $= \frac{1}{2} 3 \cdot 4 = 6$

$$\frac{1}{2} 5 \cdot h = 6 \rightarrow h = \frac{12}{5}$$

So

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{3^2} + \frac{1}{4^2} = \frac{1}{9} + \frac{1}{16} = \frac{25}{144} = \left[\frac{5}{12}\right]^2 = \left[\frac{1}{h}\right]^2$$

In vector form,

$$\left[\frac{\partial}{\partial h}\right]^2 = e^T e \quad (75)$$

where $e^T = \left[\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \cdots\right]$

From Figure 7,

$$e = B^T \bar{e} \quad (76)$$

So

$$\left[\frac{\partial}{\partial h}\right]^2 = e^T I e = [B^T \bar{e}]^T I [B^T \bar{e}] = \bar{e}^T B I B^T \bar{e} = \bar{e}^T G^{-1} \bar{e} \quad (77)$$

We use G^{-1} because $G G^{-1} = A A^T B B^T = A A^T [A^T]^{-1} B^T = A B^T = A A^{-1} = I$ (78)

Notice that equation (77) also defines an inner product. 1-Form components transform like bases vectors

so

$$\alpha^T I \alpha = \bar{\alpha}^T G^{-1} \bar{\alpha}$$

define an inner product of 1-Form components.

[1] http://www.cut-the-knot.org/pythagoras/PTForReciprocals.shtml

For two different 1-Form components - α_1 and α_2

$$\alpha_1{}^T I \alpha_2 = \bar{\alpha}_1{}^T G^{-1} \bar{\alpha}_2$$

Also in Figure 7 we can go from Node 3 to Node 1

$$\omega = IB^T \bar{e}$$

We raised the index through the Cartesian Coordinate System

$$\bar{\omega} = B\omega$$

So

$$\bar{\omega} = BIB^T \bar{e} = G^{-1} \bar{e}$$

which raises the index! – this is how we constructed the inverse metric in the previous section.

Also

$$e_i \delta^{ij} e_j = \bar{e}_i g^{ij} \bar{e}_j$$

so

δ^{ij} in Cartesian Coordinates transforms to g^{ij} in the new system.

Equation (77) in Einstein notation is written as

$$\left[\frac{\partial}{\partial h}\right]^2 = g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} \tag{79}$$

where the units $\frac{1}{m^2}$

We can do a mixed upper and lower index scenario

$$[dx + dy]\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right] = \frac{dx}{dx} + \frac{dx}{dy} + \frac{dy}{dx} + \frac{dy}{dy} \tag{80}$$

But

$$\frac{dx}{dy} = \frac{dy}{dx} = 0$$

So

(80) gives

$$\frac{dx}{dx} + \frac{dy}{dy} = 2$$

In matrix form

$$\omega^T e = \left[A^T \omega 1\right]^T B^T e1 = \omega 1^T A B^T e1 = \omega 1^T A A^{-1} e1 = \omega 1^T e1 = [dx \quad dy]\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} = \frac{dx}{dx} + \frac{dy}{dy} = 2$$

The general case for N dimensions is $\bar{\omega}^T \bar{e} = N$

$g^i{}_j = g^{in} g_{nj} = \delta^i{}_j$ matrix multiplication

$$v^i = g^i{}_j v^j$$

$$\alpha_j = g^i{}_j \alpha_i$$

$$\omega^T I e = \bar{\omega}^T I \bar{e}$$

$$\omega^j \delta^i{}_j e_i = \overline{\omega}^j \delta^i{}_j \overline{e}_i \tag{81}$$

Equation (81) shows $\delta^i{}_j$ is invariant in any coordinate system

Equation (73) - $ds^2 = \overline{\omega}^T G \overline{\omega}$ - is an example of a tensor. A tensor is a mathematical construct that allows the form of an equation to be the same in all coordinate systems.

$ds^2 = \overline{\omega}^T G \overline{\omega}$ is invariant in any coordinate system. In Cartesian coordinates, $\overline{\omega} = \begin{bmatrix} dx \\ dy \end{bmatrix}$ and $G = I$. In polar coordinates $\overline{\omega} = \begin{bmatrix} dr \\ d\theta \end{bmatrix}$ and $G = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$.

G is the metric tensor and ds^2 is the result of the metric tensor operating on $\overline{\omega}^T$ and $\overline{\omega}$ which creates a scalar. Tensors have different ranks. Tensors of rank 0 are scalars, tensors of rank 1 are ordinary vectors. Tensors of rank 2 are matrices. The metric is a 2nd rank tensor. Not all matrices are 2nd rank tensors. A 2nd rank tensor has to allow the same form of an given equation in all coordinate systems.

Figure 9 covers case where we don't start from the Cartesian coordinate system – this is an extension of Figure 7. The nodes on the right are from Figure 7

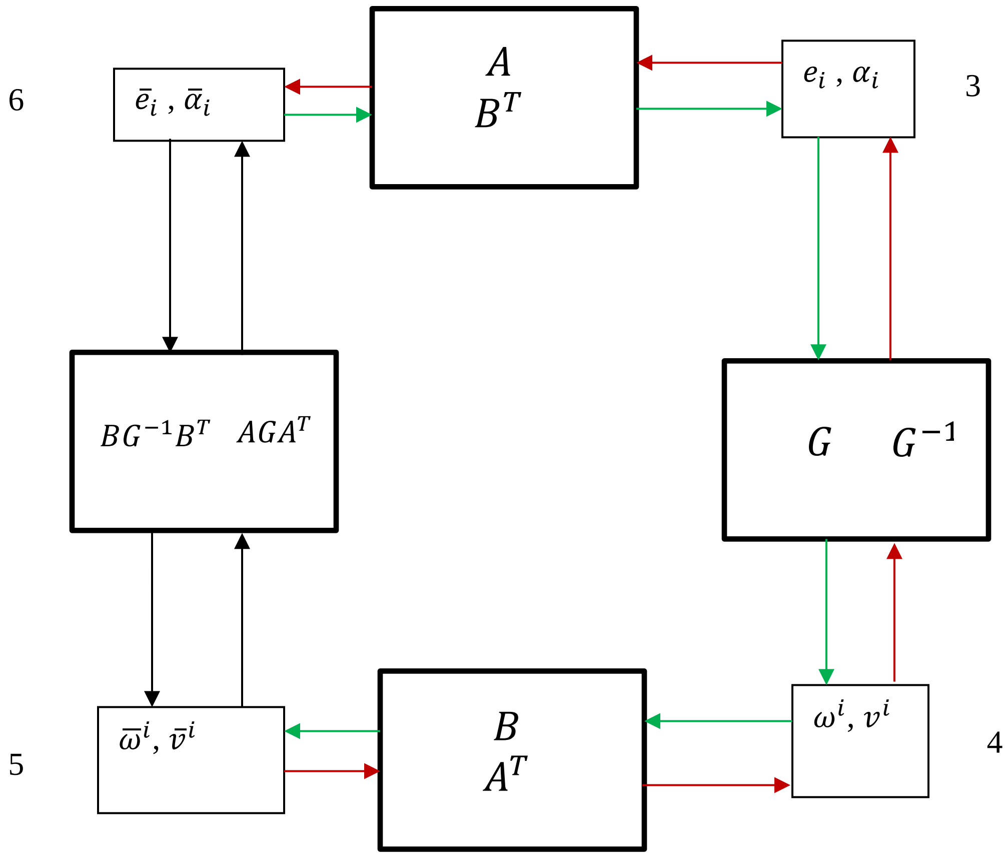


Figure 9

Figure 9 shows how G transforms to a new system using equation (82) – start at Node 5 and follow the red arrows as was shown previously.

$$\bar{G} = AGA^T \tag{82}$$

To convert it to Einstein notation take the following steps:

$$A = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \cdots \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$A^T = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \cdots \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$A^T{}_{mn} = \frac{\partial x^m}{\partial \bar{x}^n} \tag{83}$$

$$[GA^T]_{in} = g_{ij} \frac{\partial x^j}{\partial \bar{x}^n} \tag{84}$$

$$A_{qr} = \frac{\partial x^r}{\partial \bar{x}^q} \tag{85}$$

$$AGA^T = A_{qi}[GA^T]_{in} = \frac{\partial x^i}{\partial \bar{x}^q} g_{ij} \frac{\partial x^j}{\partial \bar{x}^n} = \bar{g}_{qn}$$

This is the general transformation rule for the metric and any covariant 2nd degree tensor. Covariant 2nd degree means that both indices are lower indices because they transform like the vector basis functions.

Notice that the covariant kronecker delta transforms into the metric as we saw above.

$$[IA^T]_{in} = \delta_{ij} A^T{}_{jn} = \delta_{ij} \frac{\partial x^j}{\partial \bar{x}^n}$$

$$G = AIA^T = A_{mi}[IA^T]_{in} = \frac{\partial x^i}{\partial \bar{x}^m} \delta_{ij} \frac{\partial x^j}{\partial \bar{x}^n} = g_{mn} \tag{86}$$

Figure 9 shows how G^{-1} transforms to a new system using equation (87). Start on Node 6 and follow the green arrows.

$$\bar{G}^{-1} = BG^{-1}B^T \tag{87}$$

To convert to Einstein notation, take the following steps:

$$B = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$B_{mn} = \frac{\partial \bar{x}^m}{\partial x^n} \tag{88}$$

$$B^T{}_{qr} = \frac{\partial \bar{x}^r}{\partial x^q} \tag{89}$$

$$[G^{-1}B^T]_{ir} = g^{ij}\frac{\partial \bar{x}^r}{\partial x^j} \tag{90}$$

$$BG^{-1}B^T = B_{mi}[G^{-1}B^T]_{ir} = \frac{\partial \bar{x}^m}{\partial x^i} g^{ij} \frac{\partial \bar{x}^r}{\partial x^j} = \bar{g}^{mr} \tag{91}$$

This is the general transformation rule for the metric and any contravariant 2nd degree tensor – both indices are in the numerator and transform like vector components.

Box 8

The Metric Adjusts in a Given Coordinate System
To Keep the Line Element Invariant

$$ds^2 = \omega^T I \omega = \bar{\omega}^T G \bar{\omega} \qquad \left[\frac{\partial}{\partial h}\right]^2 = e^T I e = \bar{e}^T G^{-1} \bar{e}$$

The Metric Defines Inner Products

$$v_1{}^T I v_2 = \bar{v}_1{}^T G\, \bar{v}_2 \quad \text{and} \quad \alpha_1{}^T I \alpha_2 = \bar{\alpha}_1{}^T G^{-1}\, \bar{\alpha}_2$$

The Metric Raises and Lowers Indices

$$\bar{e} = G\bar{\omega} \qquad \text{and} \qquad \bar{\omega} = G^{-1}\bar{e}$$

Metric Transformations

$$\bar{G} = AGA^T \qquad \text{and} \qquad \bar{G}^{-1} = BG^{-1}B^T$$

Non Orthogonal Coordinate System Example

The last example is a non-orthogonal coordinate system. Figure 10 shows the Cartesian coordinates with basis vectors $e = [e_1 \quad e_2]$ and basis1-Forms $\omega = [\omega^1 \quad \omega^2]$. The non-orthogonal coordinate system basis vectors are $\bar{e} = [\bar{e}_1 \quad \bar{e}_2]$.

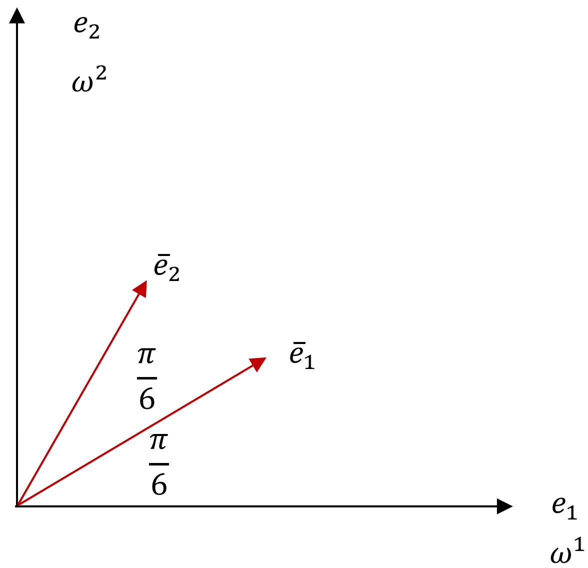


Figure 10

Equation (92) shows the Cartesian coordinates $x = [x^1 \quad x^2]$ as functions of the coordinates$\bar{x} = [\bar{x}^1 \quad \bar{x}^2]$.

$$x^1(\bar{x}^1, \bar{x}^2) = cos\left(\frac{\pi}{6}\right)\bar{x}^1 + cos\left(\frac{\pi}{3}\right)\bar{x}^2 = \frac{\sqrt{3}}{2}\bar{x}^1 + \frac{1}{2}\bar{x}^2$$

$$x^2(\bar{x}^1, \bar{x}^2) = sin\left(\frac{\pi}{6}\right)\bar{x}^1 + sin\left(\frac{\pi}{3}\right)\bar{x}^2 = \frac{1}{2}\bar{x}^1 + \frac{\sqrt{3}}{2}\bar{x}^2$$

(92)

Equation (93) shows the transformation matrix A and equation (94) shows the vector bases transformation equation.

$$A = \frac{\partial x^i}{\partial \bar{x}^j} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(93)

$$\bar{e} = Ae$$

(94)

Equations (95) and (96) show the B matrix and 1-Form bases equation.

$$B = \frac{\partial \bar{x}^i}{\partial x^j} = (A^T)^{-1} = \begin{bmatrix} \sqrt{3} & -1 \\ -1 & \sqrt{3} \end{bmatrix}$$

(95)

$$\overline{\omega} = B\omega$$

(96)

Figure 11 shows the Cartesian, $\bar{e}$, and $\overline{\omega}$ coordinate systems.

Note:

$\bar{e}_1$ and $\bar{e}_2$ are length 1

$\overline{\omega}^1$ and $\overline{\omega}^2$ are length 2

$$\bar{e}_1 \cdot \overline{\omega}^2 = \bar{e}_2 \cdot \overline{\omega}^1 = 1(2)cos\left(\frac{\pi}{2}\right) = 0$$

$$\overline{\omega}^1 \cdot \bar{e}_1 = 2(1)\cos\left(\frac{\pi}{3}\right) = \frac{2}{2} = 1$$

$$\overline{\omega}^2 \cdot \bar{e}_2 = 2(1)\cos\left(\frac{\pi}{3}\right) = \frac{2}{2} = 1$$

So

$$\bar{e}\overline{\omega}^T = e\omega^T = \frac{d\bar{e}_i}{d\bar{e}_j} = \delta^i_j = \begin{cases} 0 \; i \neq j \\ 1 \; i = j \end{cases}$$

as we would expect!

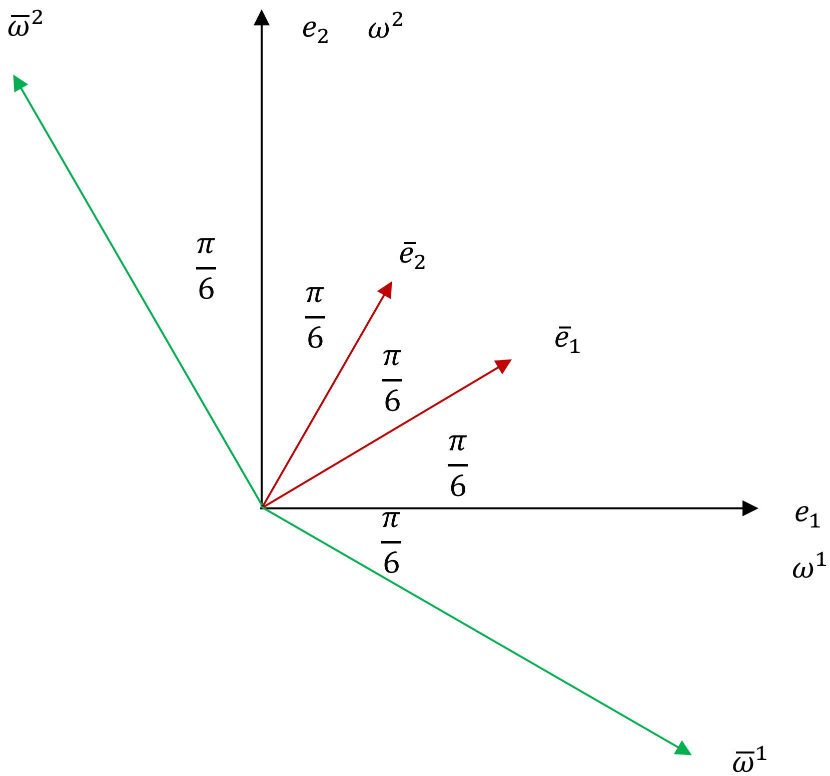


Figure 11

Equation (98) transforms the vector coordinates $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to the $\bar{x}$ system.

$$\bar{v} = Bv = \begin{bmatrix} 3^{3}/_{2} - 2 \\ 2\sqrt{3} - 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ -1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = B \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(98)

Equation (99) transforms the 1-Form coordinates $\alpha = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to the $\bar{x}$ system.

$$\bar{\alpha} = A\alpha = \begin{bmatrix} \frac{3^{3/2}+2}{2} \\ \frac{2\sqrt{3}+3}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = A \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(99)

In the Cartesian coordinate system, the inner product between α and v is shown in equation (100),

$$\alpha^T v = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 9 + 4 = 13$$

(100)

Equation (101) shows the inner product between $\bar{\alpha}$ and $\bar{v}$ in the $\bar{x}$ coordinate system.

$$\bar{\alpha}^T \bar{v} = \begin{bmatrix} \frac{3^{3/2}+2}{2} & \frac{2\sqrt{3}+3}{2} \end{bmatrix} \begin{bmatrix} 3^{3/2}-2 \\ 2\sqrt{3}-3 \end{bmatrix} = \frac{\left[3^{3/2}+2\right]\left[3^{3/2}-2\right]}{2} +$$

$$= \frac{\left[2\sqrt{3}+3\right]\left[2\sqrt{3}-3\right]}{2} = \frac{27-4}{2} + \frac{12-9}{2} = \frac{23}{2} + \frac{3}{2} = 13$$

(101)

Both Equations give the same answer as we expect.

The metric is given by equation (102)

$$G = AIA^T = g_{ij} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4}+\frac{1}{4} & \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4} & \frac{1}{4}+\frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

(102)

The path length is

$$ds^2 = [d\bar{x}_1 \quad d\bar{x}_2]\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}\begin{bmatrix} d\bar{x}_1 \\ d\bar{x}_2 \end{bmatrix} = [d\bar{x}_1 \quad d\bar{x}_2]\begin{bmatrix} d\bar{x}_1 + \frac{\sqrt{3}}{2} d\bar{x}_2 \\ \frac{\sqrt{3}}{2} d\bar{x}_1 + d\bar{x}_2 \end{bmatrix}$$

$$= d\bar{x}_1{}^2 + d\bar{x}_1 \frac{\sqrt{3}}{2} d\bar{x}_2 + d\bar{x}_2 \frac{\sqrt{3}}{2} d\bar{x}_1 + d\bar{x}_2{}^2 = d\bar{x}_1{}^2 + \sqrt{3} d\bar{x}_1 d\bar{x}_2 + d\bar{x}_2{}^2$$

$$d\bar{x}_1 = \sqrt{3} dx_1 - dx_2$$

and

$$d\bar{x}_2 = \sqrt{3} dx_2 - dx_1$$

So

$$ds^2 = d\bar{x}_1{}^2 + \sqrt{3} d\bar{x}_1 d\bar{x}_2 + d\bar{x}_2{}^2 = \left[\sqrt{3} dx_1 - dx_2\right]^2 + \sqrt{3}\left[\sqrt{3} dx_1 - dx_2\right]\left[\sqrt{3} dx_2 - dx_1\right] +$$

$$\left[\sqrt{3} dx_2 - dx_1\right]^2 = 3dx_1{}^2 - 2\sqrt{3} dx_1 dx_2 + dx^2$$

$$= -3dx_1{}^2 + 4\sqrt{3} dx_1 dx_2 - 3dx_2{}^2$$

$$= dx_1{}^2 - 2\sqrt{3} dx_1 dx_2 + 3dx_2{}^2$$

$$= dx_1{}^2 + dx_2{}^2$$

(103)

which is what we expect!