# General Coordinates

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This note discusses how to work with a general non–orthogonal coordinate system.

## Basis Vectors

A vector is specified by a directed line segment that possesses a magnitude and direction as shown in Figure 1.

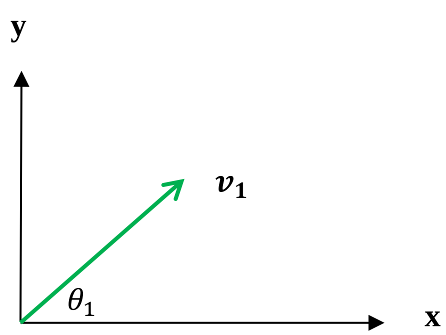



Figure 1

Figure 1 shows a vector - $\boldsymbol{v_1}$ **-** denoted by bold type -

$\boldsymbol{v_1}$ consists of a magnitude - $v_1$ – denoted by normal type and a direction angle $\theta_1$.

Figure 2 shows two non-orthogonal vectors creating a coordinate system.

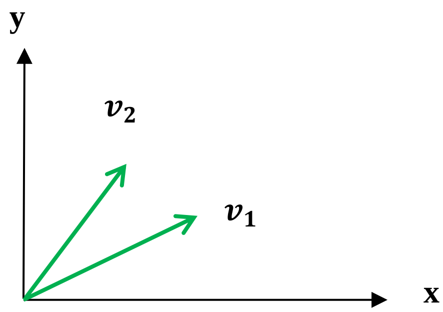



Figure 2

Table 1 shows the magnitude and direction for the coordinate system of Figure 2.

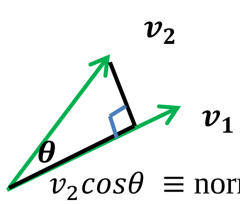
| vector | magnitude | direction | $\boldsymbol{xy}$ - components |
|---|---|---|---|
| $\boldsymbol{v_1}$ | $v_1$ | $\theta_1$ | $a_1\boldsymbol{x} + b_1\boldsymbol{y}$ |
| $\boldsymbol{v_2}$ | $v_2$ | $\theta_2$ | $a_2\boldsymbol{x} + b_2\boldsymbol{y}$ |

Table 1

To set up the general coordinate system, we need to use the dot product between two vectors.

## The Dot Product

The dot product of two vectors gives the magnitude of one vector in the direction of the other. One could think of this as a projection of one vector on to the other. Figure 3 shows the definition




$v_2 cos\theta$ ≡ normalized projection of $\boldsymbol{v_2}$ in the direction of $\boldsymbol{v_1}$

Figure 3

Dot product ≡ normalized projection * magnitude of $\boldsymbol{v_1}$

$\boldsymbol{v_1} \cdot \boldsymbol{v_2} = v_1 v_2 cos\theta$ (1)

The component form of the dot product of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ is shown in equation (2).

$\boldsymbol{v_1} \bullet \boldsymbol{v_2} = v_1 v_2 cos(\theta_2 - \theta_1) = v_1 v_2 cos\theta_2 cos\theta_1 + v_1 v_2 sin\theta_2 sin\theta_1 = a_1 a_2 + b_1 b_2$ (2)

From the property of $cos(\theta_2 - \theta_1)$

and

$x$ – components from Table 1

$v_1 cos\theta_1 = a_1$
$v_2 cos\theta_2 = a_2$

$y$ – components from Table 1

$v_1 sin\theta_1 = b_1$
$v_2 sin\theta_2 = b_2$

## Reciprocal Basis Vectors

Suppose we represent an arbitrary vector in the basis $\boldsymbol{v_1}$-$\boldsymbol{v_2}$ as shown in equation (3).

$\boldsymbol{v_3} = a\boldsymbol{v_1} + b\boldsymbol{v_2}$ (3)

for real numbers $a$ and $b$.

To recover the numbers $a$ and $b$ from $\boldsymbol{v_3}$ , we need another basis that has the properties in table 2.

| Dot product • | $\boldsymbol{\omega^1}$ | $\boldsymbol{\omega^2}$ |
|---|---|---|
| $\boldsymbol{v_1}$ | 1 | 0 |
| $\boldsymbol{v_2}$ | 0 | 1 |

Table 2

We want $\boldsymbol{\omega^i} \bullet \boldsymbol{v_j} = \boldsymbol{I}$ (4)

where $\boldsymbol{I}$ is the identity matrix – a matrix with 1's on the diagonal and zeros off the diagonal. This functions as an identity operator on vectors as shown in equation (5).

$$\boldsymbol{I} \bullet \boldsymbol{v} = \boldsymbol{v} \tag{5}$$

If we define 2 matrices

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \end{bmatrix}$$

and

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{\omega^1} \\ \boldsymbol{\omega^2} \end{bmatrix}$$

where

$\boldsymbol{e_i}$ are called basis vectors

$\boldsymbol{\omega^i}$are called reciprocal basis vectors

The matrices are a vector of row basis vectors.

For example

$\boldsymbol{e_1} = [e_{11} \quad e_{12}]$ is a row basis vector.

So equation (4) becomes

$$\boldsymbol{BA^T} = \boldsymbol{I}$$

and

$$\boldsymbol{B} = (\boldsymbol{A^T})^{-1} = \begin{bmatrix} \boldsymbol{\omega^1} \\ \boldsymbol{\omega^2} \end{bmatrix} \tag{6}$$

where

$\boldsymbol{A^T}$ is the matrix transpose operation where the rows and columns are interchanged

$\boldsymbol{A^{-1}}$ is the matrix inverse

Note also

$\boldsymbol{B} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$

To show this

$(\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}})^{\boldsymbol{T}} = \boldsymbol{I}^{\boldsymbol{T}} = \boldsymbol{I} = \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{T}}$

So

$\boldsymbol{B} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$

To complete the notation

Components of basis vectors will be labeled $v^i$ – upper index
Components of reciprocal basis vectors will be labeled $\alpha_i$ – lower index

## Numerical Example

Example vectors are shown in Table 3

| vector | $\boldsymbol{xy}$ - components |
|---|---|
| $\boldsymbol{e_1}$ | $5\boldsymbol{x} + 9\boldsymbol{y}$ |
| $\boldsymbol{e_2}$ | $7\boldsymbol{x} + 12\boldsymbol{y}$ |

Table 3

$$\boldsymbol{A} = \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix}$$

Note the basis vectors are the rows of $\boldsymbol{A}$

$$\boldsymbol{B} = (\boldsymbol{A}^{\boldsymbol{T}})^{-1} = \begin{bmatrix} -4 & \frac{7}{3} \\ 3 & \frac{-5}{3} \end{bmatrix}$$

$$\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} -4 & \frac{7}{3} \\ 3 & \frac{-5}{3} \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: The reciprocal basis vectors are the rows of $\boldsymbol{B}$

Table 4 shows the basis and reciprocal basis vectors and their components

| vector | $\boldsymbol{xy}$ - components |
|---|---|
| $\boldsymbol{e_1}$ | $5\boldsymbol{x} + 9\boldsymbol{y}$ |
| $\boldsymbol{e_2}$ | $7\boldsymbol{x} + 12\boldsymbol{y}$ |
| $\boldsymbol{\omega^1}$ | $-4\boldsymbol{x} + \frac{7}{3}\boldsymbol{y}$ |
| $\boldsymbol{\omega^2}$ | $3\boldsymbol{x} - \frac{5}{3}\boldsymbol{y}$ |

Table 4

Notice that these vectors are not normalized and don't have to be.

## The Metric

A vector represents a geometrical object whose characteristics are the same regardless of what basis set is being used – basis vectors or reciprocal basis vectors - so

$$\boldsymbol{v} = v^i \boldsymbol{e_i} = v_j \boldsymbol{\omega^j} \tag{7}$$

where there is an implied summation over repeated indices

so

$$\boldsymbol{v} \bullet \boldsymbol{e_j} = v^i \boldsymbol{e_i} \bullet \boldsymbol{e_j} = v_j \boldsymbol{\omega^j} \bullet \boldsymbol{e_j} = v_j \tag{8}$$

$\boldsymbol{G} = g_{ij} = \boldsymbol{e_i} \bullet \boldsymbol{e_j}$ is called the metric and converts a vector component - $v^i$ into its reciprocal vector component - $v_j$

From equation (7)

$$\boldsymbol{v} \bullet \boldsymbol{\omega^j} = v^i \boldsymbol{e_i} \bullet \boldsymbol{\omega^j} = \alpha_i \boldsymbol{\omega^i} \bullet \boldsymbol{\omega^j} = v^i \tag{9}$$

$\boldsymbol{G^{-1}} = g^{ij} = \boldsymbol{\omega^i} \bullet \boldsymbol{\omega^j}$ is called the inverse metric and converts a reciprocal vector component $v_i$ into its vector component $v^i$

Multiplying the metric by the inverse metric gives the identity matrix

$$g_{ij}g^{ij} = (\boldsymbol{e_i} \bullet \boldsymbol{e_j})(\boldsymbol{\omega^i} \bullet \boldsymbol{\omega^j}) = \boldsymbol{I} = \delta^i{}_j \tag{10}$$

where there is an implied summation over repeated indices

$\delta^i{}_j$ is called the Kronecker delta and is defined below – it's an identity matrix in index notation.

$$\delta^i{}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{11}$$

In matrix notation,

The metric is given by equation (12)

$$\boldsymbol{G} = g_{ij} = \boldsymbol{AA^T} \tag{12}$$

and the inverse matrix is given by equation (13)

$$\boldsymbol{G^{-1}} = g^{ij} = \boldsymbol{BB^T} \tag{13}$$

$$\boldsymbol{GG^{-1}} = g_{ij}g^{ij} = \boldsymbol{AA^T}\,\boldsymbol{BB^T}$$

but $\boldsymbol{A^T} = \boldsymbol{B^{-1}}$

so

$$\boldsymbol{GG^{-1}} = \boldsymbol{AIB^T}$$

but

$$\boldsymbol{B^T} = \boldsymbol{A^{-1}}$$

so

$$\boldsymbol{GG^{-1}} = \boldsymbol{I}$$

## Numerical Example

Table 5 shows 2 vectors in the arbitrary coordinate system from the last example

| vector | $\boldsymbol{e_1}\ \boldsymbol{e_2}$ - components | coordinates |
|---|---|---|
| $\boldsymbol{v}$ | $3\boldsymbol{e_1} + 9\boldsymbol{e_2}$ | $[v^1 = 3, v^2 = 9]$ |
| $\boldsymbol{u}$ | $\boldsymbol{e_1} + 11\boldsymbol{e_2}$ | $[u^1 = 1, u^2 = 11]$ |

Table 5

From Table 4

$$\boldsymbol{G} = \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 106 & 143 \\ 143 & 93 \end{bmatrix} \tag{14}$$

$$v_i = \boldsymbol{G}\boldsymbol{v}^{\boldsymbol{T}} = \begin{bmatrix} 106 & 143 \\ 143 & 93 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1605 \\ 2166 \end{bmatrix} \tag{15}$$

$\boldsymbol{v} = [v_i] = [1605 \quad 2166]$ in the reciprocal basis

To test, put $\boldsymbol{v}$ in $\boldsymbol{xy}$ coordinates using basis coordinates and reciprocal basis coordinates.

$$3\boldsymbol{e_1} + 9\boldsymbol{e_2} = 3\begin{bmatrix} 5 \\ 9 \end{bmatrix} + 9\begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 78 \\ 135 \end{bmatrix}$$

in $\boldsymbol{xy}$ coordinates

$$1605\boldsymbol{\omega}^{\boldsymbol{1}} + 2166\boldsymbol{\omega}^{\boldsymbol{2}} = 1605\begin{bmatrix} -4 \\ \frac{7}{3} \end{bmatrix} + 2166\begin{bmatrix} 3 \\ \frac{-5}{3} \end{bmatrix} = \begin{bmatrix} 78 \\ 135 \end{bmatrix}$$

in $\boldsymbol{xy}$ coordinates

so the reciprocal basis coordinates are correct and give the same vector in $\boldsymbol{xy}$ coordinates.

$$G^{-1} = BB^T = \begin{bmatrix} -4 & \frac{7}{3} \\ 3 & \frac{-5}{3} \end{bmatrix} \begin{bmatrix} -4 & 3 \\ \frac{7}{3} & \frac{-5}{3} \end{bmatrix} = \begin{bmatrix} \frac{193}{9} & \frac{-143}{9} \\ \frac{-143}{9} & \frac{106}{9} \end{bmatrix}$$

(16)

Test to see if reciprocal basis coordinates - $v_i$ - convert back to vector basis coordinates -$v^i$.

$$v^i = G^{-1} v_i = \begin{bmatrix} \frac{193}{9} & \frac{-143}{9} \\ \frac{-143}{9} & \frac{106}{9} \end{bmatrix} \begin{bmatrix} 1605 \\ 2166 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

(17)

which is correct.

## The Dot Product in General Coordinates

Consider two vectors in a general coordinate system.

$$\boldsymbol{u} = u^1 \boldsymbol{e_1} + u^2 \boldsymbol{e_2}$$

$$\boldsymbol{v} = v^1 \boldsymbol{e_1} + v^2 \boldsymbol{e_2}$$

The basis vectors in the general coordinate system have properties shown in Table 2 - so to get the correct orthogonality condition - we need to multiply one vector in vector coordinates by the other in the reciprocal vector coordinates as shown in equation (18).

$$\boldsymbol{u} = u^1 \boldsymbol{e_1} + u^2 \boldsymbol{e_2} = u_1 \boldsymbol{\omega^1} + u_2 \boldsymbol{\omega^2}$$

$$\boldsymbol{u} \bullet \boldsymbol{v} = u_1 v^1 \boldsymbol{\omega^1} \bullet \boldsymbol{e_1} + u_2 v^2 \boldsymbol{\omega^2} \bullet \boldsymbol{e_2} = u_1 v^1 + u_2 v^2$$

(18)

From the last section we know that metric can lower a coordinate index – go from $v^i$ to $v_i$ and the inverse metric can raise a coordinate index – go from $v_i$ to $v^i$.

So the metric and inverse metric can be used to compute a dot product in general coordinates as shown in equation (19).

$$g_{ij} v^i u^j = v^i \boldsymbol{G} \left(u^i\right)^T = g^{ij} v_i u_j = v_{\boldsymbol{i}} \boldsymbol{G^{-1}} (u_i)^T$$

(19)

where we treat $v_i$ and $u_i$ as row vectors and
$v_i$ and $u_i$ are the coordinates of the reciprocal basis vectors

## Numerical Example

Calculating $\boldsymbol{v}$ and $\boldsymbol{u}$ in Cartesian coordinates to check the dot product.

$$\boldsymbol{v} = v^i \boldsymbol{e_i} = 3\boldsymbol{e_1} + 9\boldsymbol{e_2} = 3\begin{bmatrix}5\\9\end{bmatrix} + 9\begin{bmatrix}7\\12\end{bmatrix} = \begin{bmatrix}78\\135\end{bmatrix}$$

$$\boldsymbol{u} = u^i \boldsymbol{e_i} = 1\boldsymbol{e_1} + 11\boldsymbol{e_2} = 1\begin{bmatrix}5\\9\end{bmatrix} + 11\begin{bmatrix}7\\12\end{bmatrix} = \begin{bmatrix}82\\141\end{bmatrix}$$

$$\boldsymbol{v} \bullet \boldsymbol{u} = \begin{bmatrix}78 & 135\end{bmatrix}\begin{bmatrix}82\\141\end{bmatrix} = 25431$$

Now check the dot product with the metric.

$$g_{ij}v^i u^j = v^i \boldsymbol{G}\left(u^i\right)^T = \begin{bmatrix}3 & 9\end{bmatrix}\begin{bmatrix}106 & 143\\143 & 93\end{bmatrix}\begin{bmatrix}1\\11\end{bmatrix} = 25431$$

This is correct.

Now check the dot product with the inverse metric.

$$u_i = \boldsymbol{G}\left(u^i\right)^T = \begin{bmatrix}106 & 143\\143 & 93\end{bmatrix}\begin{bmatrix}1\\11\end{bmatrix} = \begin{bmatrix}1679\\2266\end{bmatrix}$$

$$g^{ij}v_i u_j = (v_i)^{\boldsymbol{T}}\boldsymbol{G}^{-\boldsymbol{1}}u_i = \begin{bmatrix}1605 & 2166\end{bmatrix}\begin{bmatrix}\frac{193}{9} & \frac{-143}{9}\\ \frac{-143}{9} & \frac{106}{9}\end{bmatrix}\begin{bmatrix}1679\\2266\end{bmatrix} = 25431 \tag{20}$$

This is correct.