General Coordinates
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This note discusses how to work with a general non–orthogonal coordinate system.

Basis Vectors

A vector is specified by a directed line segment that possesses a magnitude and direction as shown in Figure 1.

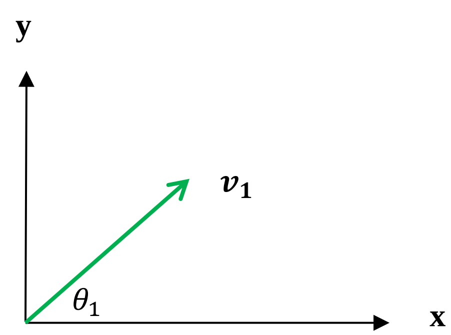


Figure 1

Figure 1 shows a vector - $\boldsymbol{v_1}$ - denoted by bold type.

$\boldsymbol{v_1}$ consists of a magnitude - v_1 – denoted by normal type and a direction angle - θ_1.

Figure 2 shows two non-orthogonal vectors creating a coordinate system.

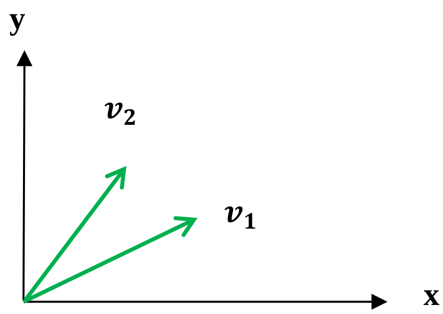


Figure 2

Table 1 shows the parameters to describe the vector – magnitude, direction and components - for the coordinate system of Figure 2.

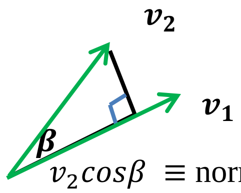
vector	magnitude	direction	$\boldsymbol{xy}$ - components
$\boldsymbol{v_1}$	v_1	θ_1	$a_1\boldsymbol{x} + b_1\boldsymbol{y}$
$\boldsymbol{v_2}$	v_2	θ_2	$a_2\boldsymbol{x} + b_2\boldsymbol{y}$

Table 1

The next section discusses the dot product between two vectors that will be used to compute the coordinates of these vectors.

The Dot Product

The dot product of two vectors gives the magnitude of one vector in the direction of the other. One could think of this as a projection of one vector onto the other. Figure 3 shows the definition.



$v_2 cos\beta$ ≡ normalized projection of $\boldsymbol{v_2}$ in the direction of $\boldsymbol{v_1}$

Figure 3

Dot product $\boldsymbol{v_1} \bullet \boldsymbol{v_2}$ ≡ (normalized projection of $\boldsymbol{v_2}$ in the direction of $\boldsymbol{v_1}$) × magnitude of $\boldsymbol{v_1}$

Equation (1) defines the dot product.

$$\boldsymbol{v_1} \cdot \boldsymbol{v_2} = v_1 v_2 cos\beta \tag{1}$$

where $\beta = \theta_2 - \theta_1$ from Table 1

The component form of the dot product of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ is shown in equation (2)[1].

$$\boldsymbol{v_1} \bullet \boldsymbol{v_2} = v_1 v_2 cos(\beta) = v_1 v_2 cos(\theta_2 - \theta_1) =$$

$$v_1 v_2 cos\theta_2 cos\theta_1 + v_1 v_2 sin\theta_2 sin\theta_1 = a_1 a_2 + b_1 b_2 \tag{2}$$

From the property of $cos(\theta_2 - \theta_1)$

and the direction of cosines of the components.

x – components from Table 1

$$v_1 cos\theta_1 = a_1$$
$$v_2 cos\theta_2 = a_2$$

y – components from Table 1

$$v_1 sin\theta_1 = b_1$$
$$v_2 sin\theta_2 = b_2$$

Also note that the dot product is linear

$$\boldsymbol{v_1} \bullet c(\boldsymbol{v_2} + \boldsymbol{v_3}) = c\boldsymbol{v_1} \bullet \boldsymbol{v_2} + c\boldsymbol{v_1} \bullet \boldsymbol{v_3}$$

where c is a constant.

[1] http://mathworld.wolfram.com/DotProduct.html

Reciprocal Basis Vectors

A note on notation - we'll adopt a notation of using $\boldsymbol{e_i}$ for basis vectors – subscripted indices. In the basis set pictured in Figure 2, $\boldsymbol{v_1}$ will be notated as $\boldsymbol{e_1}$ and $\boldsymbol{v_2}$ will be notated as $\boldsymbol{e_2}$.

Equation (3) shows an arbitrary vector in the $\boldsymbol{e_1}$ - $\boldsymbol{e_2}$ basis.

$$\boldsymbol{v_3} = a\boldsymbol{e_1} + b\boldsymbol{e_2} \tag{3}$$

for real numbers a and b.

A reciprocal basis $\boldsymbol{\omega^1}$ - $\boldsymbol{\omega^2}$ is introduced to compute the components a and b.

Table 2 shows the properties of the $\boldsymbol{\omega^1}$ - $\boldsymbol{\omega^2}$ basis needed to recover the numbers a and b from $\boldsymbol{v_3}$.

Dot product •	$\boldsymbol{\omega^1}$	$\boldsymbol{\omega^2}$
$\boldsymbol{e_1}$	1	0
$\boldsymbol{e_2}$	0	1

Table 2

Equation (4) shows this relationship.

$$\text{We want } \boldsymbol{\omega^i} \bullet \boldsymbol{e_j} = \boldsymbol{I} = \delta^i{}_j \tag{4}$$

where $\boldsymbol{I}$ is the identity matrix – a matrix with 1's on the diagonal and zeros off the diagonal. This functions as an identity operator on vectors as shown in equation (5).

$$\boldsymbol{I} \bullet \boldsymbol{v} = \boldsymbol{v} \tag{5}$$

$\delta^i{}_j$ is called the Kronecker delta and is defined below – it's an identity matrix in index notation.

$$\delta^i{}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The $\boldsymbol{\omega^i}$ vectors are called reciprocal vectors or dual vectors. The set of $\boldsymbol{\omega^i}$ reciprocal vectors is called a reciprocal basis or dual basis. Note that basis vectors are notated with a subscript index - $\boldsymbol{e_i}$ and reciprocal basis vectors are notated with a superscript index $\boldsymbol{\omega^i}$.

Now define 2 matrices

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \end{bmatrix}$$

and

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{\omega^1} \\ \boldsymbol{\omega^2} \end{bmatrix}$$

where the rows are the basis vectors.

For example

$\boldsymbol{e_1} = [e_{11} \quad e_{12}]$ is a row basis vector.

Equation (4) becomes

$$\boldsymbol{BA^T} = \boldsymbol{I}$$

so

$$\boldsymbol{B} = (\boldsymbol{A^T})^{-\boldsymbol{1}} = \begin{bmatrix} \boldsymbol{\omega^1} \\ \boldsymbol{\omega^2} \end{bmatrix} \tag{6}$$

where

$\boldsymbol{A^T}$ is the matrix transpose operation where the rows and columns are interchanged

$\boldsymbol{A^{-1}}$ is the matrix inverse

Note also

$$\boldsymbol{B} = (\boldsymbol{A^{-1}})^{\boldsymbol{T}}$$

To show this

$$(\boldsymbol{BA^T})^{\boldsymbol{T}} = \boldsymbol{I^T} = \boldsymbol{I} = \boldsymbol{AB^T}$$

So

$$\boldsymbol{B} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$$

To complete the notation

Components of basis vectors will be labeled v^i – upper index
Components of reciprocal basis vectors will be labeled α_i – lower index

Numerical Example

Example vectors are shown in Table 3

vector	$\boldsymbol{xy}$ - components
$\boldsymbol{e_1}$	$5\boldsymbol{x} + 9\boldsymbol{y}$
$\boldsymbol{e_2}$	$7\boldsymbol{x} + 12\boldsymbol{y}$

Table 3

$$\boldsymbol{A} = \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix}$$

Note the basis vectors are the rows of $\boldsymbol{A}$

$$\boldsymbol{B} = (\boldsymbol{A}^{\boldsymbol{T}})^{-\boldsymbol{1}} = \begin{bmatrix} -4 & \frac{7}{3} \\ 3 & \frac{-5}{3} \end{bmatrix}$$

$$\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} -4 & \frac{7}{3} \\ 3 & \frac{-5}{3} \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: The reciprocal basis vectors are the rows of $\boldsymbol{B}$

Table 4 shows the basis and reciprocal basis vectors and their components

vector	$\boldsymbol{xy}$ - components
$\boldsymbol{e_1}$	$5\boldsymbol{x} + 9\boldsymbol{y}$
$\boldsymbol{e_2}$	$7\boldsymbol{x} + 12\boldsymbol{y}$
$\boldsymbol{\omega^1}$	$-4\boldsymbol{x} + \frac{7}{3}\boldsymbol{y}$
$\boldsymbol{\omega^2}$	$3\boldsymbol{x} - \frac{5}{3}\boldsymbol{y}$

Table 4

Notice that these vectors are not normalized and don't have to be.

The Metric

A vector represents a geometrical object whose characteristics are the same regardless of what basis set is being used – basis vectors or reciprocal basis vectors - so

$$\boldsymbol{v} = v^i \boldsymbol{e_i} = v_j \boldsymbol{\omega^j} \tag{7}$$

where there is an implied summation over repeated indices – called Einstein notation.

so

$$\boldsymbol{v} \bullet \boldsymbol{e_j} = v^i \boldsymbol{e_i} \bullet \boldsymbol{e_j} = v_j \boldsymbol{\omega^j} \bullet \boldsymbol{e_j} = v_j = \boldsymbol{G} v^i \tag{8}$$

where

$\boldsymbol{G} = g_{ij} = \boldsymbol{e_i} \bullet \boldsymbol{e_j}$ is called the metric tensor. The metric converts a vector component - v^i into its reciprocal vector component - v_j. and will be discussed more thoroughly in a different note. Also, all combinations of the indices i and j are used so $\boldsymbol{G}$ is a matrix.

From equation (7)

$$\boldsymbol{v} \bullet \boldsymbol{\omega^j} = v^i \boldsymbol{e_i} \bullet \boldsymbol{\omega^j} = v_i \boldsymbol{\omega^i} \bullet \boldsymbol{\omega^j} = v^i = \boldsymbol{G^{-1}} v_i \qquad (9)$$

where

$\boldsymbol{G^{-1}} = g^{ij} = \boldsymbol{\omega^i} \bullet \boldsymbol{\omega^j}$ is called the inverse metric and converts a reciprocal vector component v_i into its vector component v^i

Multiplying the metric by the inverse metric gives the identity matrix

$$g_{ij} g^{jk} = (\boldsymbol{e_i} \bullet \boldsymbol{e_j})(\boldsymbol{\omega^j} \bullet \boldsymbol{\omega^k}) = \boldsymbol{I} \qquad (10)$$

again there is an implied summation over repeated indices which in this case specifies a matrix multiplication.

In matrix notation, the metric is given by equation (11)

$$\boldsymbol{G} = g_{ij} = \boldsymbol{A}\boldsymbol{A^T} \qquad (11)$$

and the inverse matric is given by equation (12)

$$\boldsymbol{G^{-1}} = g^{ij} = \boldsymbol{B}\boldsymbol{B^T} \qquad (12)$$

$$\boldsymbol{G}\boldsymbol{G^{-1}} = g_{ij} g^{jk} = \boldsymbol{A}\boldsymbol{A^T}\, \boldsymbol{B}\boldsymbol{B^T}$$

but $\boldsymbol{A^T} = \boldsymbol{B^{-1}}$

so

$$\boldsymbol{G}\boldsymbol{G^{-1}} = \boldsymbol{A}\boldsymbol{I}\boldsymbol{B^T}$$

but

$$\boldsymbol{B^T} = \boldsymbol{A^{-1}}$$

so

$$\boldsymbol{G}\boldsymbol{G^{-1}} = \boldsymbol{A}\boldsymbol{I}\boldsymbol{A^{-1}} = \boldsymbol{I}$$

Numerical Example

Table 5 shows 2 vectors in the arbitrary coordinate system from the last example

vector	$\boldsymbol{e_1}\ \boldsymbol{e_2}$ - components	coordinates
$\boldsymbol{v}$	$3\boldsymbol{e_1} + 9\boldsymbol{e_2}$	$[v^1 = 3, v^2 = 9]$
$\boldsymbol{u}$	$\boldsymbol{e_1} + 11\boldsymbol{e_2}$	$[u^1 = 1, u^2 = 11]$

Table 5

From Table 4 of the previous example

$$\boldsymbol{G} = \boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 106 & 143 \\ 143 & 93 \end{bmatrix} \tag{13}$$

$$v_i = \boldsymbol{G} v^i = \begin{bmatrix} 106 & 143 \\ 143 & 93 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1605 \\ 2166 \end{bmatrix} \tag{14}$$

$\boldsymbol{v} = v^i = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ in the vector basis

$\boldsymbol{v} = v_i = \begin{bmatrix} 1605 \\ 2166 \end{bmatrix}$ in the reciprocal basis

To test, put $\boldsymbol{v}$ in $\boldsymbol{xy}$ coordinates using basis coordinates and reciprocal basis coordinates.

$$3\boldsymbol{e_1} + 9\boldsymbol{e_2} = 3\begin{bmatrix} 5 \\ 9 \end{bmatrix} + 9\begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 78 \\ 135 \end{bmatrix}$$

$$1605\boldsymbol{\omega^1} + 2166\boldsymbol{\omega^2} = 1605\begin{bmatrix} -4 \\ \frac{7}{3} \end{bmatrix} + 2166\begin{bmatrix} 3 \\ \frac{-5}{3} \end{bmatrix} = \begin{bmatrix} 78 \\ 135 \end{bmatrix}$$

so the reciprocal basis coordinates are correct and give the same vector in $\boldsymbol{xy}$ coordinates.

$$G^{-1} = BB^T = \begin{bmatrix} -4 & \frac{7}{3} \\ 3 & \frac{-5}{3} \end{bmatrix} \begin{bmatrix} -4 & 3 \\ \frac{7}{3} & \frac{-5}{3} \end{bmatrix} = \begin{bmatrix} \frac{193}{9} & \frac{-143}{9} \\ \frac{-143}{9} & \frac{106}{9} \end{bmatrix} \tag{15}$$

Test to see if reciprocal basis coordinates - v_i - convert back to the vector basis coordinates -v^i.

$$v^i = G^{-1} v_i = \begin{bmatrix} \frac{193}{9} & \frac{-143}{9} \\ \frac{-143}{9} & \frac{106}{9} \end{bmatrix} \begin{bmatrix} 1605 \\ 2166 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} \tag{16}$$

which is correct.

The Dot Product in General Coordinates

Consider two vectors in a general coordinate system.

$$\boldsymbol{u} = u^1 \boldsymbol{e_1} + u^2 \boldsymbol{e_2}$$

$$\boldsymbol{v} = v^1 \boldsymbol{e_1} + v^2 \boldsymbol{e_2}$$

The basis vectors in the general coordinate system have properties shown in Table 2 - so to get the correct orthogonality condition - we need to multiply one vector in vector coordinates by the other in the reciprocal vector coordinates as shown in equation (17).

First put $\boldsymbol{u}$ in the reciprocal basis.

$$\boldsymbol{u} = u^1 \boldsymbol{e_1} + u^2 \boldsymbol{e_2} = u_1 \boldsymbol{\omega^1} + u_2 \boldsymbol{\omega^2} \tag{17}$$

Then take the dot product and use the linearity properties of the dot product.

$$\boldsymbol{u} \bullet \boldsymbol{v} = (u_1 \boldsymbol{\omega^1} + u_2 \boldsymbol{\omega^2}) \bullet (v^1 \boldsymbol{e_1} + v^2 \boldsymbol{e_2}) = u_1 v^1 \boldsymbol{\omega^1} \bullet \boldsymbol{e_1} + u_1 v^2 \boldsymbol{\omega^1} \bullet \boldsymbol{e_2} + u_2 v^1 \boldsymbol{\omega^2} \bullet \boldsymbol{e_1} + u_2 v^2 \boldsymbol{\omega^2} \bullet \boldsymbol{e_2}$$

But

$$\boldsymbol{\omega^i} \bullet \boldsymbol{e_j} = \boldsymbol{I} = \delta^i{}_j$$

so

$$\boldsymbol{u} \bullet \boldsymbol{v} = u_1 v^1 \boldsymbol{\omega}^{\boldsymbol{1}} \bullet \boldsymbol{e_1} + u_2 v^2 \boldsymbol{\omega}^{\boldsymbol{2}} \bullet \boldsymbol{e_2} = u_1 v^1 + u_2 v^2 \tag{18}$$

From the last section we know that the metric can lower a coordinate index – go from v^i to v_i and the inverse metric can raise a coordinate index – go from v_i to v^i.

So the metric and inverse metric can be used to compute a dot product in general coordinates as shown in equation (19).

$$u_i = \boldsymbol{G} u^i$$

$$\boldsymbol{u} \bullet \boldsymbol{v} = g_{ij} v^i u^j = \left(\boldsymbol{G} u^i\right)^{\boldsymbol{T}} \boldsymbol{v} = g^{ij} v_i u_j = (\boldsymbol{G}^{-\boldsymbol{1}} u_i)^{\boldsymbol{T}} v_{\boldsymbol{i}} \tag{19}$$

where we treat v_i and u_i as column vectors and are the coordinates of the reciprocal basis vectors.

Numerical Example

First calculate $\boldsymbol{v}$ and $\boldsymbol{u}$ in Cartesian coordinates to check the dot product.

$$\boldsymbol{v} = v^i \boldsymbol{e_i} = 3\boldsymbol{e_1} + 9\boldsymbol{e_2} = 3\begin{bmatrix}5\\9\end{bmatrix} + 9\begin{bmatrix}7\\12\end{bmatrix} = \begin{bmatrix}78\\135\end{bmatrix}$$

$$\boldsymbol{u} = u^i \boldsymbol{e_i} = 1\boldsymbol{e_1} + 11\boldsymbol{e_2} = 1\begin{bmatrix}5\\9\end{bmatrix} + 11\begin{bmatrix}7\\12\end{bmatrix} = \begin{bmatrix}82\\141\end{bmatrix}$$

$$\boldsymbol{v} \bullet \boldsymbol{u} = \begin{bmatrix}78 & 135\end{bmatrix}\begin{bmatrix}82\\141\end{bmatrix} = 25431$$

Now check the dot product with the metric computed in the last example.

$$g_{ij} v^i u^j = v^i \boldsymbol{G}\left(u^i\right)^T = \begin{bmatrix}3 & 9\end{bmatrix}\begin{bmatrix}106 & 143\\143 & 93\end{bmatrix}\begin{bmatrix}1\\11\end{bmatrix} = 25431$$

This is correct.

Now check the dot product with the inverse metric.

$$u_i = \boldsymbol{G}(u^i)^T = \begin{bmatrix} 106 & 143 \\ 143 & 93 \end{bmatrix} \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 1679 \\ 2266 \end{bmatrix}$$

$$g^{ij} v_i u_j = (v_i)^{\boldsymbol{T}} \boldsymbol{G}^{-\boldsymbol{1}} u_i = [1605 \quad 2166] \begin{bmatrix} \frac{193}{9} & \frac{-143}{9} \\ \frac{-143}{9} & \frac{106}{9} \end{bmatrix} \begin{bmatrix} 1679 \\ 2266 \end{bmatrix} = 25431 \tag{20}$$

This is correct.

Coordinate Transforms

From Table 4

$$\boldsymbol{A} = \begin{bmatrix} 5 & 9 \\ 7 & 12 \end{bmatrix}$$

But these coordinates are in terms of $\boldsymbol{x}$-$\boldsymbol{y}$ coordinates

Equation (21) shows the vector basis as a coordinate transformation.

$$\boldsymbol{e} = \boldsymbol{Ax} \tag{21}$$

In general transforming from one general basis to another is shown in equations (22) and (23) for vector and reciprocal vector basis.

$$\bar{\boldsymbol{e}} = \boldsymbol{Ae} \tag{22}$$

Reciprocal basis

$$\bar{\boldsymbol{\omega}} = \boldsymbol{B\omega} \tag{23}$$

To transform the vector components

$$\bar{\boldsymbol{v}}^{\boldsymbol{i}} = \boldsymbol{C}\boldsymbol{v}^{\boldsymbol{i}}$$

Because a vector is a geometric object, the components times the basis in one coordinate system should equal components times the basis in the other.

$$[\bar{\boldsymbol{v}}^{\boldsymbol{i}}]^{\boldsymbol{T}} \bar{\boldsymbol{e}} = [\boldsymbol{v}^{\boldsymbol{i}}]^{\boldsymbol{T}} \boldsymbol{e}$$

So

$(\boldsymbol{C}\boldsymbol{v}^{\boldsymbol{i}})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{e} = [\boldsymbol{v}^{\boldsymbol{i}}]^{\boldsymbol{T}}\boldsymbol{e}$

$[\boldsymbol{v}^{\boldsymbol{i}}]^{\boldsymbol{T}}\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{e} = [\boldsymbol{v}^{\boldsymbol{i}}]^{\boldsymbol{T}}\boldsymbol{e}$

so

$\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A} = \boldsymbol{I}$

$\boldsymbol{C} = (\boldsymbol{A}^{-\boldsymbol{1}})^{\boldsymbol{T}}$

But from equation (6)

$$\boldsymbol{B} = (\boldsymbol{A}^{\boldsymbol{T}})^{-\boldsymbol{1}} = (\boldsymbol{A}^{-\boldsymbol{1}})^{\boldsymbol{T}} \tag{6}$$

so $\boldsymbol{C}$ is the $\boldsymbol{B}$ matrix

so

$$\overline{\boldsymbol{v}}^{\boldsymbol{i}} = \boldsymbol{B}\boldsymbol{v}^{\boldsymbol{i}} \tag{24}$$

To transform the reciprocal vector components $\boldsymbol{v_i}$

$\overline{\boldsymbol{v}}_{\boldsymbol{i}} = \boldsymbol{C}\boldsymbol{v}_{\boldsymbol{i}}$

$[\overline{\boldsymbol{v}}_{\boldsymbol{i}}]^{\boldsymbol{T}}\overline{\boldsymbol{\omega}} = [\overline{\boldsymbol{v}}_{\boldsymbol{i}}]^{\boldsymbol{T}}\boldsymbol{\omega}$

$(\boldsymbol{C}\boldsymbol{v}_{\boldsymbol{i}})^{\boldsymbol{T}}\boldsymbol{B}\boldsymbol{\omega} = \boldsymbol{v}_{\boldsymbol{i}}\boldsymbol{\omega}$

so

$[\overline{\boldsymbol{v}}_{\boldsymbol{i}}]^{\boldsymbol{T}}\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{B}\boldsymbol{\omega} = [\boldsymbol{v}_{\boldsymbol{i}}]^{\boldsymbol{T}}\boldsymbol{\omega}$

so

$C^T B = I$

$B = (C^T)^{-1}$

so

$C = A$ (25)

Figure 4 is a summary of all the relationships between vectors, dual vectors, and their coordinates.

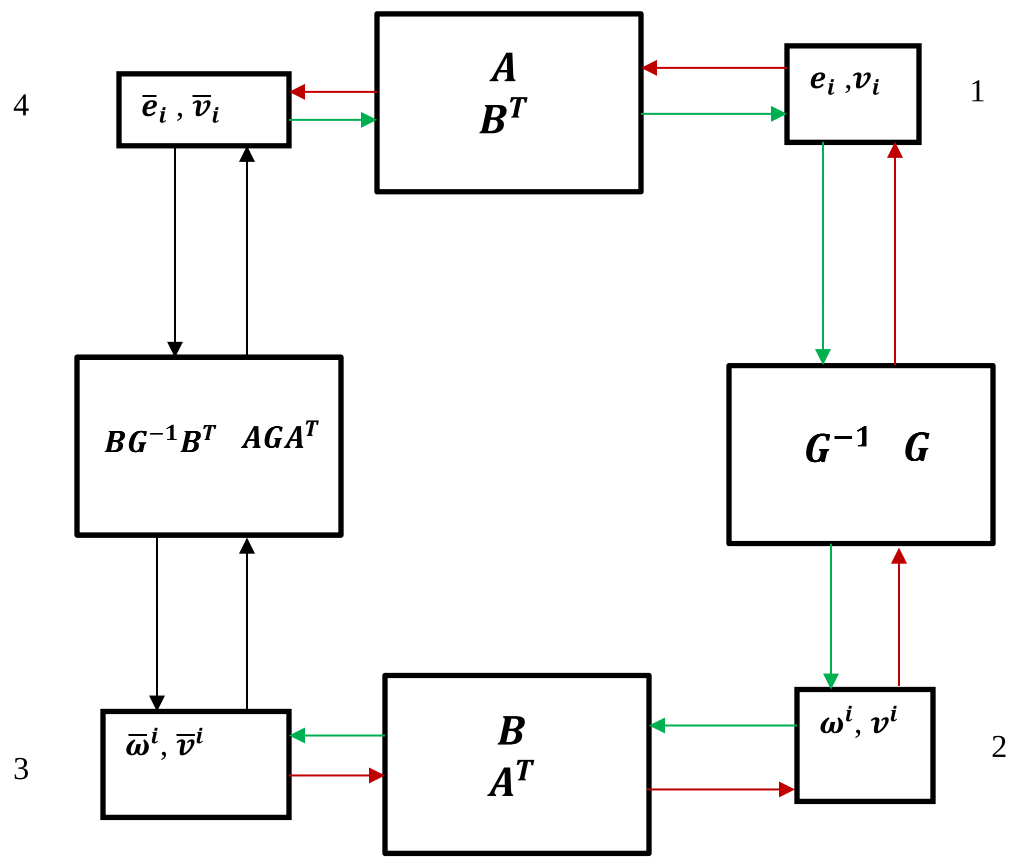


Figure 4

For example, starting at Node 3 and following the red arrows produces equation (26)

$$\overline{\boldsymbol{v}}_{\boldsymbol{i}} = \boldsymbol{AGA^T}\overline{\boldsymbol{v}}^{\boldsymbol{i}} \tag{26}$$

For $\boldsymbol{v^i}$, and $\boldsymbol{v_i}$ in Cartesian coordinates

$$\boldsymbol{G} = \boldsymbol{I}$$

and

$$\overline{\boldsymbol{v}}_{\boldsymbol{i}} = \boldsymbol{AA^T}\overline{\boldsymbol{v}}^{\boldsymbol{i}} = \overline{\boldsymbol{G}}\overline{\boldsymbol{v}}^{\boldsymbol{i}} \tag{27}$$

which is the same as equation (8) without the overlines.

$$v_j = \boldsymbol{G}v^i \tag{8}$$

Starting at Node 4 and following the green arrows we get

$$\overline{\boldsymbol{v}}^{\boldsymbol{i}} = \boldsymbol{BG^{-1}B^T}\overline{\boldsymbol{v}}_{\boldsymbol{i}} = \overline{\boldsymbol{G}}^{-\boldsymbol{1}}\overline{\boldsymbol{v}}_{\boldsymbol{i}}$$

For $\boldsymbol{v^i}$, and $\boldsymbol{v_i}$ in Cartesian coordinates

$$\boldsymbol{G^{-1}} = \boldsymbol{I}$$

$$\overline{\boldsymbol{v}}^{\boldsymbol{i}} = \boldsymbol{BB^T}\overline{\boldsymbol{v}}_{\boldsymbol{i}} = \overline{\boldsymbol{G}}^{-\boldsymbol{1}}\overline{\boldsymbol{v}}_{\boldsymbol{i}}$$

which is the same as equation (9) without the overlines

$$v^i = \boldsymbol{G^{-1}}v_i \tag{9}$$