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General Relativity without Coordinates.

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Summary. — In this paper we develop an approach to the theory of Riemannian manifolds which avoids the use of co-ordinates. Curved spaces are approximated by higher-dimensional analogs of polyhedra. Among the advantages of this procedure we may list the possibility of condensing into a simplified model the essential features of topologies like Wheeler's wormhole and a deeper geometrical insight.

1. – Polyhedra.

In this section we shall first describe our approach for the simple case of 2-dimensional manifold (surfaces). Following ALEKSANDROV [1] we develop the theory of intrinsic curvature on polyhedra. A general surface is then considered as the limit of a suitable sequence of polyhedra with an increasing number of faces. A rigorous definition of limit is not given here since it would involve a treatment of the topology on the set of all polyhedra and this would carry us too far. It is to be expected however that any surface can be arbitrarily approximated, as closely as wanted, by a suitable polyhedron. The approximation will be bad if we look at the details to the picture but an observer looking at the broad details only will find it quite satisfactory. On any surface we can define an integral Gaussian curvature by carrying out curvature experiments with geodesic triangles.

Let t be one such a triangle and let α, β, γ be its internal angles. If the geometry inside the triangle is not euclidean we have in general $\alpha+\beta+\gamma \neq \pi$.

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[1] P. S. ALEKSANDROV: *Topologia combinatoria* (Torino, 1957).

The Gaussian integral curvature ε_t of t is then defined by $\varepsilon_t = \alpha+\beta+\gamma-\pi$. On a sphere we have $\varepsilon_t = A_t/R^2$, where R is the radius of the sphere and A_t the area of t. If t shrinks to a point P and the limit $\lim_{A\to 0} \varepsilon_t/A_t = K(P)$ exists and is independent of the particular limiting procedure we take $K(P)$ as definition of local Gaussian curvature in P. On a sphere obviously $K(P) = 1/R^2$. It is well known that $\varepsilon_t = \int_t K(P)\,\mathrm{d}A$, where $\mathrm{d}A$ is the area element on the surface and the integration is carried inside t. Let it be clear however that, while the integral curvature exists under very broad conditions, the local curvature may not exist at all as an ordinary function but rather only as a measure.

This last point is essential in defining the curvature of a polyhedron M. It is obvious that if t lies entirely on a face of M we have $\varepsilon_t = 0$. The same result holds if t does not contain any vertex of M. Therefore $K(P) = 0$ if P is not a vertex.

If t contains a vertex, say V, and no other vertices, it will not depend on the explicit form of t but only on the particular choice of V. In other words $\varepsilon_t = \varepsilon_V$ is a characteristic constant of V (the deficency of V) ε_V can be found as follows. Take the sum of all internal angles of the faces of M with vertex V. This sum equals $2\pi - \varepsilon_V$. If several vertices V, X, etc., are inside t we have $\varepsilon_t = \varepsilon_V + \varepsilon_X + \varepsilon_Y + \text{etc.}$

All these results can be condensed into the original formula $\varepsilon_t = \int_t K(P)\,\mathrm{d}A$, provided one understands $K(P)$ as a Dirac type distribution, having the vertices as supports. The integral $\int_M f(P)K(P)\,\mathrm{d}A$, where $f(P)$ is a continuous function, is then to be calculated as $\sum_n f(V_n)\varepsilon_n$, where ε_n is the deficency of V_n, and the summation is carried out on all vertices of M.

If M is a compact (*i.e.* finite closed) polyhedron Gauss-Bonnet's integral curvature theorem can be written as

$$\sum_n \varepsilon_n = 2\pi(2-N)\,,$$

N is here the genus of M, $N = 0$ for a sphere and $N = 2$ for a torus. There is no loss in generality in supposing that all faces of M are triangles.

Under this form the connection of this theorem with Euler's formula for the genus is most evident. Indeed let σ_{fn} be the internal angle of the face f with the vertex n. We have:

$$\sum_f \sigma_{fn} = 2\pi - \varepsilon_n\,,$$

where the summation is carried out on all faces f having V_n as vertex. It is also clear that $\sum_n \sigma_{fn} = \pi$, where the summation now is carried out on all vertices belonging to the face f. The sum $\sum_{fn} \sigma_{fn}$ can now be carried out in two different ways by summing first on faces and using then Gauss-Bonnet's theorem or summing on vertices first. With the first method we obtain:

$$\sum_{fn} \sigma_{fn} = 2\pi v - 2\pi(2 - N) .$$

with the second

$$\sum_{fn} \sigma_{fn} = \sum_n \pi = \pi f ,$$

where v, e, f are the number of vertices, edges, faces of M respectively. Since all faces of M are triangles we have $2e = 3f$. Therefore the equation $2v - 2(2 - N) = f$ is simply Euler's formula $v - e + f = 2 - N$.

This example shows how elementary is the treatment of curvature on polyhedra.

Since we are chiefly interested in the intrinsic geometry of manyfolds we are not particularly interested in the edges of M and we regard them as a rather immaterial convention for dividing M into triangles, any other convention being just as good.

It is interesting to notice that the intrinsic geometry of M is completely fixed by the connection matrix and the length of all edges. The connection matrix is essentially a list of all faces, edges and vertices of M and a list of their mutual relationship *i.e.* by reading it one can decide which vertices, edges belong to a given face, etc. The connection matrix supplies us with all the topological information needed in the construction of M.

Once we have constructed a simplectic net on M (in plain words a division of M into triangles) the knowledge of the lengths of all edges of M implies the knowledge of all angles σ_{fn} and therefore of the deficencies ε_n.

The notion of simplectic net replaces therefore the notion of a co-ordinate net. The metric tensor on the other hand is replaced by the lengths of the edges.

If the number of vertices, faces, etc., of M increases the local Gaussian curvature will approximate a continuous function $K(P) = \varrho\varepsilon$, where ϱ is the density of vertices and ε their deficiency, provided this product varies slowly within the test triangle t.

In dealing with higher dimensional analogs of polyhedra a geodesic triangle is no longer convenient in testing curvature. We replace it with the notion of Levi-Civita's parallel transportation (shorthly LCT). Let P and Q be points of M. Let the arc a join P to Q. LCT is then an orthogonal mapping between the vector space S_p in P and the vector space S_Q in Q. If in particular $P = Q$

and a is a loop LCT maps S_p into itself. This mapping is a rotation of S around P by some angle $\varepsilon(a)$ which depends on a. By inspection it can be checked that $\varepsilon(a) = \int_a K(P)\,\mathrm{d}A$ and that therefore $\varepsilon(t) = \varepsilon_t$. ε is clearly an additive function of the loop. Take namely two loops a, b with the same end point P. Let us perform LCT around their « product » ab, ab being the loop obtained by sticking end to end the loops a and b. We get $\varepsilon(ab) = = \varepsilon(a) + \varepsilon(b)$. On a polyhedron M we shall not consider the case when a vertex lies on a loop because LCT cannot be defined unambiguously.

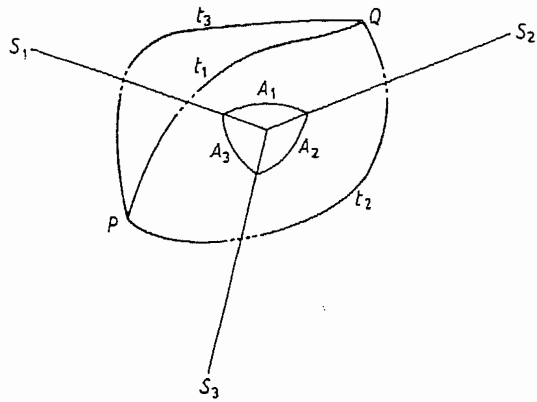

Fig. 1.

If a loop a can be deformed continuously into a loop b by keeping P fixed without encountering the above exceptional situation then $\varepsilon(a) = \varepsilon(b)$, and we write $a = b$. In the following w denotes the set of all vertices of M then $a \approx b$ is simply a homotopy in $M - w$ and is an equivalence relation. A more general and rigorous definition of homotopy will be given later. On the moment it is important to realize that the existence of a curvature set w entails the existence of equivalence classes in the set of loops with a given end point P.

2. – ε-cones.

Let us now consider a polyhedron M with one vertex only. We can visualize M as a one-sheeted cone or pyramid. There is a simple mathematical way of characterizing M. Take namely in the plane polar co-ordinates ϱ and θ so that the metric is given by $\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \varrho^2\mathrm{d}\theta^2$. In the euclidean plane it is understood that two points with the same ϱ and anomalies θ differing by a multiple of 2π are identical. We obtain a manifold with the same intrinsic geometry of a cone by replacing 2π with $2\pi - \varepsilon$.

The origin is then a vertex with deficiency ε. This manifold, hereafter named the ε-cone, is everywhere euclidean with the exception of the vertex.

These ideas can be readily extended to 3 (or higher) dimensional spaces. Let R be the real line of the variable z and let the direct product $R \cdot \varepsilon$-cone be metrized as follows:

$$ds^2 = dz^2 + d\varrho^2 + \varrho^2 d\theta^2 .$$

This space is euclidean with the exception of the straight line $\varrho = 0$. We name it the $\varepsilon - 3$-cone. If the product $R^{n-2} \times \varepsilon$-cone is considered with the metric:

$$ds^2 = dz_1^2 + dz_2^2 + \dots + dz_{n-2}^2 + d\varrho^2 + \varrho^2 d\theta^2 ,$$

we get the $\varepsilon - n$-cone. This manifold is euclidean with the exception of the $n-2$ dimensional flat subset $\varrho = 0$.

The $\varepsilon - n$-cone is the building stone of our structure. The $\varepsilon - 2$-cone is the previously defined ε-cone. If the dimension n is otherwise known we shall drop it from the writing.

3. – Fundamental group.

Take now a path in a Riemannian n-dimensional manifold M_n. We define a path p as a point-valued continuous function $P(s)$ of a parameter s, $0 \leqslant s \leqslant 1$, $P(0)$, $P(1)$ being the end point of the path. If $P(0) = P(1) = P$ then the path is called a loop. We shall regard as identical the loops $P(s)$ and $P(s'(s))$ where $s'(s)$ is non decreasing and continuous and $s(0) = 0$, $s(1) = 1$. If we have two paths a, b, defined by the functions $A(s)$, $B(s)$ such that $A(1) = B(0)$ then the product $c = ab$ is defined by the function $C(s) = A(2s)$ if $s \leqslant 0.5$, $C(s) = B(2s-1)$ if $s \geqslant 0.5$. The product $c = ab$ joins $A(0)$ to $B(1)$ through $A(1) = B(0)$. This product is associative but in general is not commutative. If a, b are loops with the same end point P then ab and ba both exist and are loops with end point P. In general, however, $ab \neq ba$. a^{-1}, the reverse of a, is defined by $A(1-s)$. Suppose now that our manifold is almost everywhere euclidean except a closed subset w which carries curvature, to be described in detail later. Take two paths a, b with the same end points, defined by $A(s)$ and $B(s)$. We always suppose $B(s) \notin w$, $A(s) \notin w$. If there is a point-valued function $P(s, t)$ of s, $0 \leqslant s \leqslant 1$, and t, $0 \leqslant t \leqslant 1$, continuous in both s and t, such that $P(s, 0) = A(s)$, $P(s, 1) = B(s)$ we say that we have deformed a into b, and the function $P(s, t)$ is called a deformation. If for given a, b it is possible to find a deformation such that $P(s, t) \notin w$ for all s, t then we write $a \sim b$. The symbol $\sim$ satisfies the formal properties of an equivalence (holonomy). Indeed $a \sim a$ because one such a deformation is given by $P(s, t) = A(s)$. If $a \sim b$ then $P(s, 1-t)$ deforms b into a and $P(s, 1-t) \notin w$ it follows $a \sim b$. Similarly if $P(s, t)$ deforms a into b and $Q(s, t)$ deforms b into c and therefore $a \sim b$, $b \sim c$, the function $R(s, t) = P(s, 2t)$ if $t < 0.5$ and $R(s, t) = Q(s, 2t-1)$ if $t > 0.5$ deforms a into c and $R(s, t) \notin w$, $a \sim c$. These definitions apply as well to loops, with the additional requirement that loops remain such during the deformation or that $P(0, t) = P(1, t)$ for every t. If moreover we have $P(0, t) = P(1, t) = P(0, 0) = P$ then we write $a \approx b$. Clearly $a \approx b$ implies $a \sim b$ but the converse is not true. The symbol $\approx$ applies to loops only.

The symbol $\approx$ also defines an equivalence (homotopy in $M_n - w$). The proof is trivial. If we only know that $a \sim b$ then generally the function $C(s) = P(0, s) = P(1, s)$ defines a loop c because $P(0, 0) = P(0, 1) = P$. Surprisingly enough we have b $b \approx c^{-1}ac$. This relation can be checked from the deformation $S(s, t)$ defined as:

$$\begin{aligned} S(s, t) &= C(1-3ts) & s < \tfrac{1}{3}\,, \\ &= P(3s-1, 1-t) & \tfrac{1}{3} < s < \tfrac{2}{3}\,, \\ &= C(1+3(s-1)t) & \tfrac{2}{3} < s < 1\,. \end{aligned}$$

We define next the unit loop u in the point P through the function $U(s) = P = \text{const}$. Clearly $aa^{-1} \approx u$. Suppose now that $a \approx a'$ and $b \approx b'$. Let $P(s, t)$ deform a into a' and $Q(s, t)$ deform b into b'. The function $R(s, t) = P(2s, t)$, $(s < 0.5)$ and $R(s, t) = Q(2s-1, t)$, $(s > 0.5)$ is then a deformation with $R(0, t) = R(1, t) = p$ of ab into $a'b'$ and $R(s, t) \notin w$. Therefore from $a \approx a'$ and $b \approx b'$ it follows $ab \approx a'b'$.

Take now a set of loops with end point P. This set can be partitioned into equivalence classes. A class is identified by one element belonging to it and from now on, unless differently stated, we use a loop to label the whole class. If a and b are classes then all products $a'b'$ where $a \approx a'$ and $b \approx b'$ are homotopic and belong to a new class which we define as the product of the classes a and b and write ab. Moreover if a is a class and u the unit class then $au = a = ua$.

To any class a we can associate the inverse class a^{-1} and $aa^{-1} = a^{-1}a = u$. The product of classes is associative. We see therefore that the set of all classes satisfies the axioms of a group, the fundamental group. Now to each loop a, we can associate an orthogonal matrix $S(a)$ as follows. Take a vector A in P and carry A around a according to LCT. There will be a linear, norm-preserving, mapping between the initial and final position of A. This mapping can be therefore represented by an orthogonal matrix if we choose in P a local orthogonal frame of reference. Of course, as we shall require in the applications, if our space is locally pseudoeuclidean we shall have Lorentz matrices. Now the important point is that homotopic loops correspond to the same matrix or, otherwise said, the orthogonal matrix is a function of classes. This in turn implies that the mapping classes matrices are a representation of the fundamental group on the vector space of P. To prove this we first notice that all loops $\approx u$ correspond to unit matrices. If $v \approx u$ then there is a deformation function $T(s, t)$ which carries v into u. The set of all points

$T(s,t)$ is a simply connected surface Σ and we can choose on Σ and in a suitable neighborhood of Σ a globally euclidean frame of reference and the result of L-C-transportation along v calculated in this frame is the identity.

It is furthermore evident that if $c = ab$ and if the matrices $S(a)$, $S(b)$. $S(c)$ correspond to a, b, c we have $S(c) = S(b)\,S(a)$. Take namely a vector V in P and let us carry it along a. The result is by definition $S(a)V$. We then carry this vector along b. We obtain $S(b)\,S(a)V$. But in so doing we have carried V along c, therefore $S(v)V = S(b)\,S(a)V$ and the result follows. If then $a \approx a'$ we know that $a = va'$ where $v \approx u$. It follows $S(a) = S(v)\,S(a')$ but $S(v) = 1$ and $S(a) = S(a')$. The variable in $S(a)$ is then understood to be an equivalence class. The fundamental group has in general infinitely many elements. The example of the ε-cone is instructive. If we take a loop in the ε-cone we can give the angle θ of a point on the loop as function of s. Let $\theta(0) = 0$ without loss of generality. We know that $\theta(1) = N(2\pi - \varepsilon)$. N gives the number of times the loop has encircled the line $\varrho = 0$ or bone, which coincides with the subset w. Loops with the same N are clearly homotopic. Let $a(N)$ be the equivalence class with $\theta(1) = N(2\pi - \varepsilon)$. We have $a(N)\,a(M) = a(N+M)$. The fundamental group is then isomorphic to the group of integers under addition. In order to find $S(a(n))$, shortly $S(n)$ and $S(0) = S$, we notice that $S(N) = S^N$. S will be called the generator of the bone. Take then a vector V in P, and carry it along $a(1)$. V can be split into orthogonal components, one lying into the $\varepsilon - 2$-cone, the other into R^{n-2}. The latter is unaffected by the process, but the first is rotated by the angle ε. The total effect is that of an orthogonal $n \times n$ matrix with R^{n-2} as invariant subspace. In a $\varepsilon - 3$-cone, R^{n-2} is just a real line orthogonal to an $\varepsilon - 2$-cone. S is a rotation with axis $R^{n-2} = R$. This particular fundamental group is abelian but, in general, this is not true.

4. – We shall define now n-dimensional generalizations of polyhedra, shortly skeleton spaces, where curvature is carried by a $n-2$ subset w, the skeleton. One such structure is the $\varepsilon - n$-cone.

We start from a simplectic decomposition of an n-dimensional space M_n. This decomposition determines the topology of M_n but not the metric. We define a metric in M_n wtih the following axioms.

A) The metric in the interior of any n-dimensional closed simplex T_n is euclidean. This means that we can calculate the distance of any two points inside T_n, define on T_n a cartesian system of co-ordinates, and give the co-ordinates of the points of the boundary of T_n in this frame.

B) In the metric of T_n the boundary of T_n is decomposable into the sum of $n+1$ closed simplexes T_{n-1} and these simplexes are flat.

C) If a simplex T_{n-1} is common boundary of T_n and $T'_{n'}$ the distance of any two points of T_{n-1} is the same in both frames of T_n and $T'_{n'}$.

D) If $P \in T_n$, $P' \in T'_n$ and P, P' are close enough to T_{n-1} we define the distance PP' as the infimum of $PQ+QP'$ for all $Q \in T_{n-1}$.

With this definition the metric is euclidean also on T_{n-1} except perhaps on T_{n-2}'s on the boundary of T_{n-1}. Our definition works on a strictly local scale only but this is all we need to join smoothly neighbouring simplexes. If we want to introduce indefinite metrics a similar but more complicated definition can be used with the same results. Our manifold is now everywhere euclidean outside the union w of all T_{n-2} boundary simplexes (bones). In a M_3 the bones are straight segments while the T_{n-3} are points. Each bone connects two T_0's. If the length of the bones is given, the geometrical structure of the simplexes and of M_3 is determined and in particular their internal dihedral angles can be calculated from elementary formulas of spherical trigonometry. If there would be no curvature on a bone, then the sum of all dihedral angles around the bone should be 2π. But if these lengths are chosen at random, this sum for the i-th bone will be $2\pi - \varepsilon_i$. Here we recognize that the cross section of such a bone is locally an $\varepsilon_i - 2$-cone and that therefore locally the bone is an $\varepsilon_i - 3$-cone. In a S_n the T_{n-2} will carry along locally the same geometry of an $\varepsilon - n$-cone. Take now a T_0 in M_3. In general this point is common end of several bones $T_1^1 \dots T_1^m$. We orient the bones in such a way as to have the positive direction outgoing from T_0. We refer then to T_0 as to an oriented m-joint.

In an m-joint there is a remarkable identity among the $S_p p: 1 \dots m$, S_p being the generator of the p-th bone. In order to find it we may deal with an idealized isolated joint, where all departing bones stretch up to infinity. We then select a particular cyclic order of the bones, for instance the one defined by the index p, (0 follows m) the particular choice being irrelevant. We consider then the set of plane sectors, A_p having two contiguous bones T^p, T^{p+1} as sides. We suppose that no two such sectors have common points except perhaps a common boundary bone. These sectors then cut the space in two regions M' and M''. Select the test point P in one of these regions, say M', and a point Q in M''. Take now m paths t_p, joining P to Q, and therefore intersecting the boundary between M' and M''. Let t_p intersect A_p only and only once. Now $t_p t_{p-1}^{-1}$ is a loop a_p with end point P encircling T_1^p once. a_p is a suitable candidate as generator of T_1^p. It must be pointed out however that any other $a'_p \sim a_p$ is just as good and that our procedure merely selects unambiguously one such a_p. We have than the fundamental identity:

$$(1) \qquad \begin{cases} a_1 a_2 a_3 \dots a_p \approx u \,, \\ \text{or} \\ S_1 S_2 \dots S_p = 1 \,. \end{cases}$$

The latter identity binds the deficiencies ε_p of the bones and their mutual angles. This can be clearly seen if $m = 3$. In this case:

$$S_1 S_2 S_3 = 1 .$$

This identity can be visualized with the help of a theorem of Euler which states that if in a spherical triangle the sides are in the order the angles of $A_1 A_2 A_3$, then the opposite internal angles are $\pi - \varepsilon_3$, $\pi - \varepsilon_2$, $\pi - \varepsilon_1$. Therefore if we know the deficiency we know the mutual angles of the bones and viceversa.

As we shall see later, in our geometrical approach (1) plays the role of Bianchi's identity in a differentiable Riemannian manifold.

There are corresponding generalizations of (1) in higher dimensional spaces. More precisely there is one such identity at each T_{n-3}, binding the generators of all T_{n-2} bones having this T_{n-3} as boundary. Not all these identities are independent since it is enough to write them for all T_{n-3} having a given T_{n-4} as boundary but one, this one being a consequence of the others. In a sense a T_{n-4} acts as a negative constraint by diminishing the number of Bianchi identities. We have not carried our investigation as far as to show what the actual number of Bianchi's identities is and in particular what is the role of T_{n-5} etc., in determining it since the problem does not arise in M_4.

5. – Differentiable manifolds.

The transition of a skeleton space into a differentiable manifold can be accomplished if we increase the density ϱ of bones while at the same time keeping $\varrho\varepsilon$ slowly varying. We have already shown how this procedure works on a surface. Here we shall proceed in similar fashion on a M_n, $n > 2$. To begin with let's deal with a boundle of parallel bones on a M_3. We suppose that the curvature induced by the bones is small so that we may regards M_3 as approximately euclidean. Let $\boldsymbol{U}$ be a unit vector parallel to the bones. We test the curvature by carrying a vector $\boldsymbol{A}$ around a small loop of area Σ and unit normal $\boldsymbol{n}$, $\boldsymbol{\Sigma} = \Sigma\boldsymbol{n}$. At the end of the test $\boldsymbol{A}$ is found to have rotated around $\boldsymbol{U}$ by the angle $\sigma = N\varepsilon$, where N is the number of bones entangled in the loop. We have $N = \varrho(\boldsymbol{U}, \boldsymbol{\Sigma})$, being the number of bones passing through the loop $\boldsymbol{\Sigma} = \boldsymbol{U}$. The final result is then:

$$\boldsymbol{A} = \varrho\varepsilon(\boldsymbol{U} \wedge \boldsymbol{A})(\boldsymbol{U}, \boldsymbol{\Sigma}) .$$

On the other hand from standard textbooks we know that:

$$\begin{aligned} A_\mu &= R^{\sigma}_{\varrho\cdot\alpha\beta}\, \Sigma^{\alpha\beta} A_\sigma , \\ \Sigma^{\alpha\beta} &= \varepsilon^{\alpha\beta\gamma}\Sigma_\gamma , \\ \varepsilon^{\alpha\beta\gamma} &= \pm 1 = -\varepsilon^{\beta\alpha\gamma} , \quad \text{etc.} \end{aligned}$$

By comparing these results we get:

$$R_{\mu\sigma\,\alpha\beta} = \varrho\varepsilon U_{\mu\sigma} U_{\alpha\beta}\,; \qquad U_{\mu\sigma} = \varepsilon_{\mu\sigma\lambda} U^{\lambda}\,. \tag{2}$$

This approximation has all the symmetry properties of the full Riemann tensor. If several bundles are present their effects at this stage are merely additive. In M_n a bone has an orientation determined by a skew symmetric tensor $U_{\mu\sigma}$, $U_{\mu\sigma}U^{\mu\sigma} = 2$, $U_{\mu\sigma}U_{\alpha\beta} + U_{\mu\alpha}U_{\beta\sigma} + U_{\mu\beta}U_{\sigma\alpha} = 0$. If we deal with a bundle of parallel bones we may chose co-ordinates x_1, x_2 perpendicular to the bone, $x_3 \dots x_n$ parallel to it. In this case $U_{12} = -U_{21} = 1$, while all other components vanish. Eq. (2) still holds provided μ, σ, α, β are allowed to take all values $1 \dots n$. Independently of n the scalar curvature R is given by $2\varrho\varepsilon$. We shall now show the connection between eq. (1) and Bianchi's identities in this approximation. If the ε's are small (1) can be written as:

$$\sum_1^m \varepsilon_p U^p_{\mu\sigma} = 0 \quad \text{or} \quad \sum_1^m \varepsilon_p U^p = 0\,; \qquad \text{for an } m\text{-joint in } M_3.$$

Suppose next that there is a distribution of identical m-joints of density ϱ. This means also that we have m bundles of parallel bones, the p-th bundle having defect ε_p, direction U^p, density ϱ_p. Of course ϱ_p cannot be considered constant because an m-joint produces a decay or creation (depending on the direction) of a bone into or from the others. The rate of decay depends on ϱ. This rate can be estimated as follows. By definition ϱ_p is the density of flux of the outgoing bundle of bones through a surface Σ orthogonal to $\boldsymbol{U}_p$.

Let the position of Σ be determined by an abscissa s computed along the direction of $\boldsymbol{U}_p$. Let C be a cylinder of height $\mathrm{d}s$ and basis $\Sigma(s)$. Each joint in C increases by a unit the flux of ϱ_p bones through $\Sigma(s+\mathrm{d}s)$. There are $\Sigma\cdot\varrho\cdot\mathrm{d}s$ joints in C and therefore:

$$\Sigma\,[\varrho_p(s+\mathrm{d}s) - \varrho_p(s)] = \Sigma\,\varrho(s)\,\mathrm{d}s \qquad \text{or} \qquad \frac{\mathrm{d}\varrho_p}{\mathrm{d}s} = \varrho\,,$$

it follows

$$(\boldsymbol{U}_p\,\mathbf{grad}\,\varrho_p) = \varrho\,.$$

The Riemann tensor is then given by eq. (2) as:

$$R_{\alpha\beta\,\lambda\delta} = \Sigma_p \varepsilon_p \varrho_p U^p_{\alpha\beta} U^p_{\lambda\delta}\,. \tag{3}$$

In our approximation covariant derivatives reduce to normal derivatives. Bianchi's identities can be written as:

$$B_{\alpha\beta;\lambda\delta\varepsilon} = \frac{\partial}{\partial x_\varepsilon} R_{\alpha\beta,\lambda\delta} + \frac{\partial}{\partial x_\lambda} R_{\alpha\beta,\delta\varepsilon} + \frac{\partial}{\partial x_\delta} R_{\alpha\beta,\varepsilon\lambda} = 0\,. \tag{4}$$

We introduce (3) into (4) and treat ε_p and U_p as constants. Use in made of operator identity:

$$U_{\lambda\delta}\frac{\partial}{\partial x_\varepsilon}+U_{\delta\varepsilon}\frac{\partial}{\partial x_\nu}+U_{\varepsilon\lambda}\frac{\partial}{\partial x_\delta}=(\boldsymbol{U},\,\mathbf{grad})\varepsilon_{\nu\lambda\delta}\,,$$

the result is then:

$$B_{\alpha\beta.\lambda\delta\varepsilon}=\sum_p(\boldsymbol{U}_p,\,\mathbf{grad}\,\varrho_p)\varepsilon_p U^p_{\alpha\beta}\varepsilon_{\lambda\delta\varepsilon}=\varepsilon_{\lambda\delta\varepsilon}\varrho\sum_p\varepsilon_p U^p_{\alpha\beta}=0\,,$$

by virtue of (1). The connection between (1) and Bianchi's identities is then evident.

6. – Lorentz manifolds.

If the metric of a skeleton space is not positive definite then we must replace orthogonal matrices with Lorentz matrices. Correspondingly the defects may become imaginary. Also the usual triangular inequalities among sides of a simplex cease to hold, for instance in a triangle where all sides are timelike one side is larger than the sum of the others. In general, however, it is evident how to modify ordinary trigonometry in order to fit the needs. In an M_4 bones are triangles and, if the metric is indefinite with one time co-ordinate, there are three kinds of bones:

1) Spacelike. Every vector on the bone is spacelike. It is imaginary.

2) Null. Every vector on the bone is a linear combination of a null vector and a spacelike vector orthogonal to it. In this case the corresponding Lorentz matrices cannot be diagonalized. We have $\varepsilon=0$.

3) Timelike. There are both timelike and spacelike vectors on the bone. ε is real.

For simplicity we shall restrict ourselves to timelike bones only or at most null bones.

A useful parameter in the following is the area L of the bone. Let A_μ, B_μ be two sides of the bone. We define L as follows:

$$4L^2=(A_\mu B^\mu)^2-(A_\mu A^\mu)(B_\mu B^\mu)\,.$$

L is therefore always a real quantity. If the bone is timelike, L is real and we choose always $L>0$. We restrict ourselves to timelike bones also in order to avoid ambiguities in the definition of L. If the bone is null then $L=0$.

7. – Variational principles.

The most direct way of approximating an Einstein space with a skeleton space is to use a variational principle. The lagrangian is evidently

$$\mathscr{L} = \frac{1}{16\pi}\int R\,\mathrm{d}^4x\,\sqrt{-g}\,,$$

calculated on the skeleton and we vary this lagrangian in the lengths of all T_1: This choice is particularly happy because we know that these lengths yield the same type of information as the metric tensor. Moreover there is a simple expression for $\mathscr{L}$ in terms of the deficiencies and areas af the bones. Indeed we know that within the T_4 and T_3 there is no contribution to the integral because R is a distribution with the support w. L is an additive function of the bones. Let n label the bones in the skeleton. Then $\mathscr{L} = \sum_n F(T_2^n)$. The function F is of course the same function for all bones. Since the bone is homogeneous, $F(T_2^n)$ is proportional to the area of the bone $F(T_2^n) = L_n f(\varepsilon_n)$, where f depends on ε_n only. T_2^n can be considered as the superposition of two bones $T_2^{n'}$ and $T_2^{n''}$ with the same shape and area and such that $\varepsilon_n = \varepsilon_n' + \varepsilon_n''$. It follows $f(\varepsilon_n) = f(\varepsilon_n') + f(\varepsilon_n'')$. The latter condition implies $f(\varepsilon) = C\varepsilon$, where C is a constant. We may write then $\mathscr{L} = C\sum_n \varepsilon_n L_n$. This result has to be compared with $\mathscr{L} = (1/8\pi)\int \varepsilon\varrho\,\mathrm{d}^4x$ in the limit discussed in Section 5 and for a bundle of identical bones of equal area. From the comparison we get $C = 1/8\pi$ and $\mathscr{L} = (1/8\pi)\sum_n L_n\varepsilon_n$. In M_m the corresponding result is $\mathscr{L} = C\sum_n \varepsilon_n L_n^m$, where L_n^m is the $m-2$ dimensional measure of T_{m-2}^m.

The next task is then the variation of $\mathscr{L}$ in the lengths l_p of T_1^p: While it is clear how L_n depends on l_p, the dependence of ε_n on l_p is rather involved. However quite remarkably we can carry out the variation of $\mathscr{L}$ as if the ε_n were constants:

$$\delta\mathscr{L} = \frac{1}{8\pi}\sum_n \varepsilon_n\,\delta L_n\,,$$

(See the Appendix for the proof; the result holds in any dimension.)

In an M_3 the measures L_p^3 are simply the lengths l_p of T_1^p, which are all independent, the field equations are then simply $\varepsilon_n = 0$, *i.e.* all 3 dimensional Einstein skeletons are flat, as expected.

In M_4 the field equations are instead:

$$\sum_n \varepsilon_n\,\frac{\partial L_n}{\partial l_p} = 0\,.$$

The summation is now extended on all n such that l_p is a side of L_n. We have then:

$$\frac{\partial L_n}{\partial l_p} = l_p \operatorname{ctg} \theta_{pn} ,$$

θ_{pn} being the angle which is opposite to l_p in T_2^n. Finally:

$$\sum_n \varepsilon_n \operatorname{ctg} \theta_{pn} = 0 \tag{5}$$

are Einstein's equations for an empty skeleton space. They can be shown, with the methods of Section 5, to degenerate into $R_{\mu\nu} = 0$ for a differential manifold.

If M_4 is a compact manifold, and the number of simplexes in the decomposition is finite, say $1 \leqslant p \leqslant P$, we have P equations (5) in P variables l_p. One such variable can be however fixed at will since if l_p is a solution of (5), also λl_p is a solution. In other words (5) is a set of P homogeneous equations in the l_p's and in general we should have no solution to (5) except the trivial $l_p = 0$. However we have no proof that one or more eq. (5) are not independent of the others and the question of the existence of a compact Einstein skeleton space is still open. Any answer to the question in our simplified model would bring strong evidence in favor of a similar answer for differentiable manifolds. Also (5) is formally suitable for relaxation methods or it could be programmed on a computer if the question of the existence of a solution is first met.

Appendix

We shall show now that for infinitesimal variations of l, we have in M_m

$$\sum_p \delta\varepsilon_p L_p^m = 0 , \tag{6}$$

L_p^m is the $m-2$ dimensional measure of the n-th bone.

Let T_n be a simplex and let in T_n be defined a cartesian set of co-ordinates. T_n has $n+1$, T_{n-1}^r boundary simplexes, $r:1 \dots n+1$. Any two simplexes T_{n-1}^r, T_{n-1}^s have a common boundary T_{n-2}^{rs}. Let U_μ^r be the unit vector normal to T_{n-1}^r, $r:0 \dots n-1$, and $U_\mu^r U^{r,\mu} = 1$. If θ_{rs} is the angle between T_{n-1}^r and T_{n-1}^s we have:

$$\cos \theta_{rs} = U_\mu^r U^{s,\mu} .$$

Next we consider the skew symmetric tensor

$$U_{\mu\lambda}^{rs} = - U_{\lambda\mu}^{rs} ; \sin \theta_{rs} U^{rs} = U_\mu^r U_\lambda^s - U_\lambda^r U_\mu^s .$$

Clearly we have $U^{rs}_{\lambda\mu} U^{rs,\lambda\mu} = 2$ so that $U^{rs}_{\lambda\mu}$ is exactly the tensor used in Section 5, eq. (2) and following, and defines the orientation of T^{rs}_{n-2}. If L_{rs} is the measure of T^{rs}_{n-2} we have the following identity:

(7)
$$\sum_s L_{rs} U^{rs} = 0 :$$

take namely a frame where $U^r_0 = 1$, $U^r_{\mu \neq 0} = 0$. Then $V^s_\lambda = U^{rs}_{\lambda 0}$, $\lambda \neq 0$ are the only non vanishing components of $U^{rs}_{\lambda\mu}$ and $V^s_\lambda V^{s,\lambda} = 1$. In T^r_{n-1} the quantity $\sum L_{rs} V^s_\lambda A^\lambda$ is the flux of the constant vector A^λ through the boundary of T^r_{n-1} and therefore vanishes.

Take now the formula:

$$\delta\theta_{rs} = \frac{-1}{\sin\theta_{rs}} \left(U^r_\mu \delta U^{s,\mu} + U^s_\mu \delta U^{r,\mu} \right) ,$$

and transform it using the identity:

$$U^{rs}_{\lambda\mu} U^{r,\lambda} = \frac{1}{\sin\theta_{rs}} \left(\cos\theta_{rs} U^r_\mu - U^s_\mu \right) ,$$

and:

$$U^{rs}_{\lambda\mu} U^{r,\lambda} \delta U^{r,\mu} = - \frac{1}{\sin\theta_{rs}} \left(U^s_\mu \delta U^{r,\mu} \right) ,$$

having used $U^r_\mu \delta U^{r,\mu} = 0$. We get:

$$\delta\theta_{rs} = U^{rs}_{\lambda\mu} \left(U^{r,\lambda} \delta U^{r,\mu} + U^{s,\lambda} \delta U^{s,\mu} \right) ,$$

so that by (7):

$$\sum_{r,s} L_{rs} \delta\theta_{rs} = 0 .$$

(6) then follows by taking the sum of the above identity on all simplexes of the skeleton.

* * *

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RIASSUNTO

In questo lavoro viene sviluppato il formalismo matematico della relatività generalizzata in modo da evitare l'uso di coordinate. A tal uopo si introducono decomposizioni simpliciali di varietà rimanniane che costituiscono l'analogo iperdimensionale di poliedri. Si spera che questo nuovo formalismo renda possibile la discussione di soluzioni delle equazioni di Einstein corrispondenti a topologie di elevata complessità.