The Arithmetic Geometric Mean (AGM) Algorithm

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The Arithmetic Geometric Mean (AGM) algorithm provides a very efficient and very accurate way to evaluate elliptic integrals. The AGM algorithm is as follows:

$$a_{n+1} = \frac{a_n + b_n}{2}$$

$$b_{n+1} = \sqrt{a_n b_n} \tag{1}$$

The algorithm converges when $|a_n - b_n| < \varepsilon$ (2)

where ε is a specified error tolerance

M(a,b) – see reference 1 - is the limiting process of the AGM i.e. $\lim_{n\to\infty} |a_n - b_n| = 0$

A complete elliptical integral of the first kind is defined as

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{3}$$

The relationship of the AGM to an Elliptical Integral of the first kind is specified as

$$\frac{1}{M(1,x)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - x^2)\sin^2 \theta}} \tag{4}$$

See reference 2

If we let $k^2 = 1 - x^2$ the right hand side of (4) = $K(k)$

The Elliptical Integral of the first kind can be written in terms of a modular angle as follows:

$$K(\alpha) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}}$$

(5)

In equation (5), $k^2 = \sin^2\alpha$, therefore, $x^2 = \cos^2\alpha$ using the relation $k^2 = 1-x^2$. The complete Elliptic Integral of the first kind with a modular angle can be evaluated by equation (6)

$$\frac{1}{M(1,\cos\alpha)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}} \Rightarrow \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}} = \frac{\pi}{2}\frac{1}{M(1,\cos\alpha)} \qquad (6)$$

Here are some examples of the AGM to compute $K(\alpha)$

Example 1.)

$$\alpha = 15° \rightarrow \alpha = \frac{\pi}{180} \times 15 = 0.261799387799149$$

N	a_n	b_n	$\lvert a_n - b_n \rvert$
0	1	0.965925826289068	0.0340741737109317
1	0.982962913144534	0.982815255421419	0.000147657723115535
2	0.982889084282976	0.982889081510181	2.77279543769993e-009
3	0.982889082896579	0.982889082896579	0

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}} = \frac{\pi}{2}\frac{1}{M(1,\cos\alpha)} = \frac{\pi}{2}\frac{1}{0.982889082896579} = 1.59814200211254$$

This is accurate to 15 decimal places. See reference 3.
Note: 32 decimal place precision – 16 decimal place displayed - was used in Maxima – see reference 4 - for the AGM implementation.

This shows the power of the AGM - it converges to 15 decimal places in 3 iterations!

Example 2.)

$$\alpha = 30^\circ \;\rightarrow\; \alpha = \frac{\pi}{180} \times 30 = 0.523598775598299$$

N	a_n	b_n	$\lvert a_n - b_n \rvert$
0	1	0.86602540378444	0.13397459621556
1	0.93301270189222	0.9306048591021	0.0024078427901197
2	0.93180878049716	0.93180800274782	7.7774934126306761e-007
3	0.93180839162249	0.93180839162241	8.1157303100098943e-014
4	0.93180839162245	0.93180839162245	0

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha \sin^2\theta}} = \frac{\pi}{2}\frac{1}{M(1,\cos\alpha)} = \frac{\pi}{2}\frac{1}{0.93180839162245} = 1.685750354812596$$

and is accurate to 15 decimal places in 4 iterations! See reference 3.

Example 3.)

$$\alpha = 35^\circ \;\rightarrow\; \alpha = \frac{\pi}{180} \times 35 = 0.81915204428899$$

N	a_n	b_n	$\lvert a_n - b_n \rvert$
0	1	0.81915204428899	0.18084795571101
1	0.9095760221445	0.90507018749321	0.0045058346512873
2	0.90732310481885	0.9073203077754	2.7970434527402332e-006
3	0.90732170629713	0.90732170629605	1.077804512306102E-012
4	0.90732170629659	0.90732170629659	0

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha \sin^2\theta}} = \frac{\pi}{2}\frac{1}{M(1,\cos\alpha)} = \frac{\pi}{2}\frac{1}{0.90732170629659} = 1.731245175657058$$

which is accurate to 15 decimal places in 4 iterations! See reference 3.

References:

1.) Jonathan M. Borwein, Peter B. Borwein, “Pi and the AGM”, John Wiley & Sons, Inc, pp 1-2 (1987)
2.) Jonathan M. Borwein, Peter B. Borwein, “Pi and the AGM”, John Wiley & Sons, Inc, page 5 (1987)
3.) Abramowitz and Stegun, “Handbook of Mathematical Functions”, Dover, pp 610 – 611 (1972)
4.) http://maxima.sourceforge.net/