The Arithmetic Geometric Mean (AGM) Algorithm
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The Arithmetic Geometric Mean (AGM) algorithm provides a very efficient and very accurate way to evaluate elliptic integrals. The AGM algorithm is as follows:

\[ a_{n+1} = \frac{a_n + b_n}{2} \]

\[ b_{n+1} = \sqrt{a_n b_n} \] (1)

The algorithm converges when \( |a_n - b_n| < \varepsilon \) (2)

where \( \varepsilon \) is a specified error tolerance

\( M(a,b) \) – see reference 1 - is the limiting process of the AGM i.e. \( \lim_{n \to \infty} |a_n - b_n| = 0 \)

A complete elliptical integral of the first kind is defined as

\[ K(k) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \] (3)

The relationship of the AGM to an Elliptical Integral of the first kind is specified as

\[ \frac{1}{M(1,x)} = \frac{2}{\pi} \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - x^2) \sin^2 \theta}} \] (4)

See reference 2

If we let \( k^2 = 1 - x^2 \) the right hand side of (4) = \( K(k) \)

The Elliptical Integral of the first kind can be written in terms of a modular angle as follows:
In equation (5), \( k^2 = \sin^2 \alpha \), therefore, \( x^2 = \cos^2 \alpha \) using the relation \( k^2 = 1 - x^2 \). The complete Elliptic Integral of the first kind with a modular angle can be evaluated by equation (6)

\[
\frac{1}{M(1, \cos \alpha)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \alpha \sin^2 \theta}} \Rightarrow \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \alpha \sin^2 \theta}} = \frac{\pi}{2} \frac{1}{M(1, \cos \alpha)}
\]

(6)

Here are some examples of the AGM to compute \( K(\alpha) \)

Example 1.)

\( \alpha = 15^\circ \rightarrow \alpha = \frac{\pi}{180} \times 15 = 0.261799387799149 \)

| N | \( a_n \) | \( b_n \) | \( |a_n - b_n| \) |
|---|---|---|---|
| 0 | 1 | 0.965925826289068 | 0.0340741737109317 |
| 1 | 0.982962913144534 | 0.982815255421419 | 0.000147657723115535 |
| 2 | 0.982889084282976 | 0.982889081510181 | 2.77279543769993e-09 |
| 3 | 0.982889082896579 | 0.982889082896579 | 0 |

\[
\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \alpha \sin^2 \theta}} = \frac{\pi}{2} \frac{1}{M(1, \cos \alpha)} = \frac{\pi}{2} \frac{1}{0.982889082896579} = 1.59814200211254
\]

This is accurate to 15 decimal places. See reference 3.

Note: 32 decimal place precision – 16 decimal place displayed - was used in Maxima – see reference 4 - for the AGM implementation.

This shows the power of the AGM - it converges to 15 decimal places in 3 iterations!
Example 2.)

\[ \alpha = 30^\circ \rightarrow \alpha = \frac{\pi}{180} \times 30 = 0.523598775598299 \]

\[
\begin{array}{|c|c|c|c|}
\hline
N & a_n & b_n & |a_n - b_n| \\
\hline
0 & 1 & 0.86602540378444 & 0.13397459621556 \\
1 & 0.93301270189222 & 0.9306048591021 & 0.0024078427901197 \\
2 & 0.93180878049716 & 0.93180800274782 & 7.775934126306761e-07 \\
3 & 0.93180839162249 & 0.93180839162241 & 8.115730310009894e-014 \\
4 & 0.93180839162245 & 0.93180839162245 & 0 \\
\hline
\end{array}
\]

\[
\frac{\pi}{2} \int_0^\frac{\pi}{2} \frac{d\theta}{\sqrt{1 - \sin^2 \alpha \sin^2 \theta}} = \frac{\pi}{2} \frac{1}{M(1, \cos \alpha)} = \frac{\pi}{2} \frac{1}{2 \cdot 0.93180839162245} = 1.685750354812596
\]

and is accurate to 15 decimal places in 4 iterations! See reference 3.

Example 3.)

\[ \alpha = 35^\circ \rightarrow \alpha = \frac{\pi}{180} \times 35 = 0.81915204428899 \]

\[
\begin{array}{|c|c|c|c|}
\hline
N & a_n & b_n & |a_n - b_n| \\
\hline
0 & 1 & 0.81915204428899 & 0.18084795571101 \\
1 & 0.9095760221445 & 0.90507018749321 & 0.0045058346512873 \\
2 & 0.90732310481885 & 0.9073203077754 & 2.7970425742032e-06 \\
3 & 0.90732170629605 & 0.90732170629605 & 1.07780451206102E-012 \\
4 & 0.90732170629659 & 0.90732170629659 & 0 \\
\hline
\end{array}
\]

\[
\frac{\pi}{2} \int_0^\frac{\pi}{2} \frac{d\theta}{\sqrt{1 - \sin^2 \alpha \sin^2 \theta}} = \frac{\pi}{2} \frac{1}{M(1, \cos \alpha)} = \frac{\pi}{2} \frac{1}{2 \cdot 0.90732170629659} = 1.731245175657058
\]

which is accurate to 15 decimal places in 4 iterations! See reference 3.

References:

3.) Abramowitz and Stegun, “Handbook of Mathematical Functions”, Dover, pp 610 – 611 (1972)
4.) http://maxima.sourceforge.net/